NATIONAL MATHEMATICS MAGAZINE

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VOL. XIX | BATON ROUGE, LA., February, 1945 | No. 5

Entered as second-class matter at Baton Rouge, Louisiana.
Published monthly excepting June, July, August, and September, by S. T. SANDERS.

THIS JOURNAL IS DEDICATED TO THE FOLLOWING AIMS: (1) Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values. (2) To supply an additional medium for the publication of expository mathematical articles. (3) To promote more scientific methods of teaching mathematics. (4) To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Every paper on technical mathematics offered for publication should be submitted (with enough enclosed postage to cover two two-way transmissions) to the Chairman of the appropriate Committee, or to a Committee member whom the Chairman may designate to examine it, after being requested to do so by the writer.

Papers intended for a particular Department should be sent to the Chairman of that Department.

Mathematicians and Their Inspirations

Tartaglia was inspired to a solution of the general cubic equation by his preparation for a contest in problem solving. The challenge came from Antonio del Florido, who knew only an empirical solution of $x^3+qx=r$. Tartaglia solved all the problems, Florido solved none.

Ferrari—famed for his solution of the fourth degree general equation—found his ability in mathematics by listening to Cardan's lectures while serving him as an errand boy.

Vieta—as in Tartaglia's case—was inspired to some of his best work by a problem-solving challenge.

Again it was Descartes' success in responding to a challenge offered to the world to solve a certain geometrical problem that decided him to abandon the army and devote his life to mathematics.

Simpson, the English mathematician, had his first interest in mathematics aroused by a study of the eclipse of the sun in 1724.

Lagrange's serious attention to mathematics was occasioned by a reading of the astronomer Halley's memoir "On the Excellence of the Modern Algebra in Certain Optical Problems". Inspired by this tract Lagrange studied mathematics so diligently that at the end of a year, according to the historian Ball, he was "an accomplished mathematician".

At the age of 13, William Rowan Hamilton discovered by accident a copy of Newton's Universal Arithmetic. It was the reading of this that inspired him to study analysis. He became the founder of the theory of quaternions.

According to David Brewster, Isaac Newton did not begin seriously to study at school until one day when he suffered a painful blow from a school-fellow who ranked above him. Newton's retaliation took the form of an intense desire to outrank his enemy. This he did in time, by hard study.

Pascal's initial interest in mathematics was especially whetted by a summary order from his father to have nothing to do with mathematical books. To enforce his order the father kept such books concealed from the son. One day young Pascal asked his parent to tell him what mathematics was about. The answer was that "it was the method of making figures with exactness, and finding out what proportions they relatively had to one another!" This answer inspired the boy to ponder in secret on the nature and properties of the various geometric figures. At the age of 16 he wrote a masterly treatise on the conics.—Editorial in *Mathematics News Letter*, November, 1932.

S. T. Sanders.

A Geometrical Interpretation of the Invariant System of Two Binary Cubics*

By Kathryn B. Rolfe
University of California

1. *Introduction.* It has been proved† that if two triangles are inscribed in a conic, their sides touch a second conic, called the involution conic, which is also touched by the sides of ∞^1 other inscribed triangles. If a parameter t is spread along the base conic, the vertices of a triangle will be determined by the roots of a binary cubic equation f. Thus associated with the ∞^1 triangles is a pencil of cubics $f_1+\lambda f_2$ which define a cubic involution, $I_{1,2}$, in which one point of a set determines the other two. There are, in general, four double points given algebraically by the Jacobian of the two cubics or geometrically by the contacts on the base conic of the common tangents of the two conics. After finding the equation of the involution conic, we wish to examine its relations to the invariant system of the two binary cubics.

2. *The equation of the involution conic.* Taking the base conic in the canonical form

$$(1) \qquad x_1=t^2, \quad x_2=2t, \quad x_3=1;$$

and the two cubics with roots t_1, t_2, t_3, and t_1', t_2', t_3', we find, by considering the conic on five of the six sides of the triangles, the line equation of the involution to be

$$(2) \qquad (s_3\sigma_2-\sigma_3s_2)u_1^2+4(s_3-\sigma_3)u_2^2+(s_1-\sigma_1)u_3^2+2(s_3\sigma_1-\sigma_3s_1)u_1u_2$$
$$+(s_2\sigma_1-\sigma_2s_1+\sigma_3-s_3)u_1u_3+2(s_2-\sigma_2)u_2u_3=0,$$

where s_i and σ_i are the symmetric functions of t_1, t_2, t_3 and t_1', t_2', t_3', respectively. Now, if the cubics have the form (3) $a_0y^3+a_1t^2+$

$$(3) \qquad a_0t^3+a_1t^2+a_2t+a_3=0 \quad \text{and} \quad a_0't^3+a_1't^2+a_2't+a_3'=0,$$

s_i and σ_i may be expressed in terms of the coefficients. Substituting in Equation (2), and letting

$$p_{ij}=\begin{vmatrix} a_i & a_i' \\ a_j & a_j' \end{vmatrix},$$

*Abstract of a master's thesis, written at the University of Washington under the direction of R. M. Winger.

†Emil Weyr, "ueber Involutionen höherer Grade", Journal für die Reine und Angewandte Mathematik, Vol. 72, pp. 285-292, 1870.

the conic assumes the form

$$p_{23}u_1^2+4p_{03}u_2^2+p_{01}u_3^2+2p_{31}u_1u_2+(p_{12}+p_{30})u_1u_3+2p_{20}u_2u_3=0. \tag{4}$$

3. *Condition for a degenerate involution conic.* We are now ready to study the invariant system of the cubics. First, suppose the two cubics have a common root. This means the two triangles inscribed in the base conic have a common vertex. Since a conic cannot have four tangents to it from a point, it seems that the involution conic must degenerate into two points. This can be proved analytically, for the condition that two cubics have a common root is that the resultant vanish, and the condition that the conic degenerate into two points is that the discriminant of its line equation vanish.

Remembering that $p_{ij}=-p_{ji}$, and noting that $p_{01}p_{23}+p_{02}p_{31}+p_{03}p_{12}\equiv 0$, the discriminant of the conic is found to be

$$\nabla=p^3_{30}+2p_{03}p_{01}p_{23}+p_{02}p_{30}p_{31}-p^2_{20}p_{23}-p_{01}p^2_{31}+p_{01}p_{12}p_{23}. \tag{5}$$

Now, Salmon* has found this to be the negative of the resultant of the two cubics. Therefore, we have the

Theorem. *The necessary and sufficient condition that the involution conic degenerate into two points is that the two cubics have a common root.*

The cubic $I_{1,2}$ now degenerates to $(t-t_1)(q_1+\lambda q_2)$ where q_1 and q_2 are quadratics. The sets now all contain the point $t_1\equiv t_1'$ and the pairs of the quadratic involution, which are cut out by the pencil of lines on the second point of the involution conic.

4. *Verification of* $\Theta^2-4\Delta\Theta'=0$. It is interesting to verify that it is possible to inscribe a triangle in the base conic which will be circumscribed about the involution conic. The invariant relation on two conics that such a triangle may be constructed is $\Theta^2-4\Delta\Theta'=0$,† where f is the point equation of the inscribed conic, ϕ the point equation of the circumscribed conic, Δ and Δ' the respective discriminants of their point equations, Θ is the polar of f with respect to ϕ, and Θ' is the polar of ϕ with respect to f. By the polar of f with respect to ϕ we mean the result obtained by replacing u_i in the line equation of f by $\partial/\partial x_i$, and operating on ϕ.

If the conics have the point equations

$$f:\quad a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2+2a_{23}x_2x_3+2a_{31}x_3x_1+2a_{12}x_1x_2=0$$

$$\phi:\quad b_{11}x_1^2+b_{22}x_2^2+b_{33}x_3^2+2b_{23}x_2x_3+2b_{31}x_3x_1+2b_{12}x_1x_2=0,$$

*George Salmon, *Modern Higher Algebra*, Fourth Edition, p. 77.
†Salmon, *Conic Sections*, p. 342.

their line equations are

$$F:\quad A_{11}u_1^2+A_{22}u_2^2+A_{33}u_3^2+2A_{23}u_2u_3+2A_{31}u_3u_1+2A_{12}u_1u_2=0$$

$$\Phi:\quad B_{11}u_1^2+B_{22}u_2^2+B_{33}u_3^2+2B_{23}u_2u_3+2B_{31}u_3u_1+2B_{12}u_1u_2=0,$$

where A_{ij} is the cofactor of a_{ij} in the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \qquad a_{ij}=a_{ji}.$$

Θ is then $A_{11}b_{11}+A_{22}b_{22}+A_{33}b_{33}+2A_{23}b_{23}+2A_{31}b_{31}+2A_{12}b_{12}$, and Θ' is $a_{11}B_{11}+a_{22}B_{22}+a_{33}B_{33}+2a_{23}B_{23}+2a_{31}B_{31}+2a_{12}B_{12}$.

If we let f be our involution conic, and ϕ be our base conic in points, we readily write down ϕ and Φ and we have F to within a factor from Equation (4). To determine this factor, we find f from Equation (4), and from f derive F.

After performing the calculations, we have

$$\begin{aligned} f:\quad & (p^2_{20}-4p_{03}p_{01})x_1^2+(p^2_{12}/4+p_{12}p_{30}/2+p^2_{30}/4-p_{23}p_{01})x_2^2 \\ & +(p^2_{31}-4p_{23}p_{03})x_3^2+(2p_{01}p_{31}-p_{20}p_{12}-p_{20}p_{30})x_1x_2 \\ & +(4p_{03}p_{12}-4p^2_{03}-2p_{31}p_{20})x_1x_3+(2p_{20}p_{23}-p_{12}p_{31}-p_{30}p_{31})x_2x_3=0, \end{aligned}$$

$$\phi:\quad x_2^2-4x_1x_3=0,$$

$$\begin{aligned} F:\quad & \nabla[p_{23}u_1^2+4p_{03}u_2^2+p_{01}u_3^2+2p_{31}u_1u_2+(p_{12}+p_{30})u_1u_3 \\ & +2p_{20}u_2u_3]=0, \end{aligned}$$

$$\Phi:\quad -4u_2^2+4u_1u_3=0,\quad \Delta:\ -\nabla^2,\quad \Delta':\ -4,\quad \Theta:\ 6\nabla P,\quad \Theta':\ -9P^2,$$

where $P=p_{03}-p_{12}/3$.

Hence, $\Theta^2-4\Delta\Theta'=36\nabla^2P^2-4\nabla^2\cdot 9P^2=0$, which verifies that a triangle can be constructed such that it is inscribed in ϕ and circumscribed about f.

5. *Condition for apolar conics.* Θ and Θ' are invariants of the conics, and when either vanishes, the conics are said to be apolar. A geometrical interpretation of Θ is*

The necessary and sufficient condition that a triangle can be constructed (a) inscribed in ϕ and self-polar to f or (b) circumscribed to f and self-polar to ϕ is that $\Theta=0$.

A similar statement may be made for the vanishing of Θ'.

Salmon† has calculated the two combinants P and Q of our cubics

*R. M. Winger, *Projective Geometry*, p. 285.

†George Salmon, *Modern Higher Algebra*, Fourth Edition, pp. 204-206.

and shown that p^3-27Q is the resultant R of the cubics, which we in turn have identified with $-\nabla$ of the conics.

$$P=p_{03}-p_{12}/3, \tag{6}$$

$$-Q=p^3_{12}/729+p^2_{20}p_{23}/27+p^2_{13}p_{01}/27-p_{01}p_{12}p_{23}/27 \tag{7}$$
$$-p_{03}p^2_{12}/81-p_{03}p_{01}p_{23}/9.$$

$P=0$ is the condition that the cubics f_1 and f_2 be apolar, and $Q=0$ is the condition that it be possible to determine λ so that $f_1+\lambda f_2$ shall be a perfect cube. How are these invariants related to the conics?

If $P=0$, we note that both Θ and Θ' vanish, which means the base conic and the involution conic are mutually apolar. Also, it is possible to either inscribe in or circumscribe to ϕ a triangle self-polar to f, and to either inscribe in or circumscribe to f a triangle self-polar to ϕ. Thus, we have the

Theorem. *If the cubics f_1 and f_2 are apolar, the conics are mutually apolar, and conversely.*

6. *Condition that the conics have contact.* Since the properties of the involution may be studied by taking any two cubics of the pencil $f_1+\lambda f_2$, let us take f_1 to be at^3+bt^2 and f_2 to be $c't+d'$. This fixes one double point at O and another at ∞. Then, noting that $p_{01}=p_{23}=0$, and substituting the values of u_1, u_2, and u_3 from the parametric line equations of the base conic

$$u_1=1, \quad u_2=-t, \quad u_3=t^2$$

In Equation (4), we have

$$2p_{02}t^3+(p_{12}+3p_{03})t^2+2p_{13}t=0, \tag{8}$$

which gives the parameters of the common lines of the two conics, or of the double points of the involution. That is, the left member of Equation (8) is the Jacobian of the base cubics.

We now have

$$Q=p^2_{12}(9p_{03}-p_{12})/729, \tag{9}$$

and can discover its relation to the conics. We will first prove that the vanishing of Q is a sufficient condition that the conics have contact.

Since Q vanishes if either $p_{12}=0$ or $p_{12}=9p_{03}$, let us examine the two cases.

Case A: $p_{12}=0$. In order that $p_{12}=0$, either $b=0$ or $c'=0$, and hence either $p_{02}=0$ or $p_{13}=0$. If $p_{02}=0$, Equation (8) becomes $3p_{03}t^2+2p_{13}t=0$, which means two double points coincide at $t=\infty$. If $p_{13}=0$,

Equation (8) becomes $2p_{02}t^3+3p_{03}t^2=0$, which means two double points coincide at $t=0$. Hence if $p_{12}=0$, the two conics have contact.

Case B: $p_{12}=9p_{03}$. In this case, Equation (8) reduces to $t(p_{02}t^2+6p_{03}t+p_{13})=0$, with roots at $t=0$, $t=\infty$, and where the discriminant of the quadratic is $9p^2_{03}-p_{02}p_{13}$. Now, if $p_{12}=9p_{03}$, then $bc'=9ad'$, and $p_{02}p_{13}=ac'bd'=9a^2d'^2=9p^2_{03}$, and the discriminant equals zero. Two double points, therefore, coincide at $t=-3p_{03}/p_{02}$.

Hence, the vanishing of Q is a sufficient condition that the conics have contact. Let us see whether the condition is necessary.

If we have two concics f and ϕ, the discriminant of $kf+\phi$ is $\Delta k^3+\Theta k^2+\Theta' k+\Delta'$. That is, the roots of $\Delta k^3+\Theta k^2+\Theta' k+\Delta'=0$, considered as a cubic in k, give the values of k for which $k_i f+\phi=0$ $(i=1, 2, 3)$ are the equations of the three pairs of lines through the four intersection points of the involution conic with the base conic. If the cubic has a double root, two pairs of lines will be identical, and, therefore, the conics will have contact. Let us find the condition that this happen. Substituting the values for Δ, Θ, Θ', and Δ' in the cubic, the equation becomes

$$\nabla^2k^3-6\nabla Pk^2+9P^2k+4=0, \tag{10}$$

whose discriminant reduces to $16\nabla^3(\nabla+P^3)$. Since $27Q-P^3=\nabla$, the discriminant becomes, neglecting a numerical factor, ∇^3Q, which vanishes if $\nabla=0$ or if $Q=0$. If $\nabla=0$, we have the special case in which the involution conic degenerates into two points. Hence we have the

THEOREM. $Q=0$ *is a sufficient condition that the base conic and the involution conic have contact, but is a necessary condition only if the involution conic does not degenerate.*

If $Q\neq 0$, but $\nabla=0$, the conics may be said to have contact. One point will then be on the base conic, and the tangent at this point will count for two common tangents. The other two are then the tangents from the second point to the base conic.

7. *Condition for double-contact conics.* If the conics have two contacts, the Jacobian giving the double points of the involution must have two square factors. Now it is well known that this happens when the involution is based on a cubic and its cubicovariant.* Taking one cubic as t^3-1, its cubicovariant is t^3+1 and the involution conic then reduces to $4u_2^2-u_1u_3=0$, which touches the base conic at $t=0$, ∞. The converse may be proved for a proper involution conic. For pro-

*Winger, *Projective Geometry*, p. 219.

jectively, double contact conics are equivalent to concentric circles. If then the base conic is taken as the unit circle in absolute coordinates, the system of triangles in which the involution conic is inscribed will all be equilateral. Among these triangles will be two whose vertices are given by t^3-1 and t^3+1, i. e., a cubic and its cubicovariant, which proves the theorem.

If the involution conic degenerates into two points on the base conic, we have a sort of improper double contact. Thus if the conic reduce to $u_1u_3=0$, all p_{ij} vanish except p_{12} and we may write the base cubics of the involution in the canonical form $f_1=t_1^2t_2$ and $f_2=t_1t_2^2$, which have two common roots. In this case neither cubic is the cubicovariant of the other but the Jacobian is $t_1^2t_2^2$. We may now summarize as follows:

*A necessary and sufficient condition that a proper involution conic have two distinct contacts with the base conic is that the cubic involution be based on a cubic and its cubicovariant. But if the general sets in the involution have two common and distinc points, the involution conic breaks up into these points on the base conic, and conversely, while the involution does not contain a cubic and its cubicovariant.**

8. *Condition that the conics osculate.* To determine the condition that the conics osculate, let us return to the cubics $f_1=at^3+bt^2$ and $f_2=c't+d'$. Suppose that $P=Q=0$. Then $p_{12}=3p_{03}$, and, therefore, in either Case A or Case B, $p_{12}=p_{03}=0$. This means that $bc'=0$ and $ad'=0$. If $a=b=0$ or $c'=d'=0$, we have the case in which one cubic and the equation of the involution conic vanish identically. If $a=c'=0$, Equation (8) becomes $p_{13}t=0$, which has three roots at $t=\infty$ and one at $t=0$. If $b=d'=0$, Equation (8) becomes $p_{02}t^3=0$, which has three roots at $t=0$ and one at $t=\infty$. Therefore, if $P=Q=0$, the conics have 3-point contact. That is, they osculate.

But, from the relation $\nabla=27Q-P^3$, we find that in this case the involution conic degenerates into two points. How are these points situated? The equation of the conic becomes

$$p_{31}u_1u_2+p_{20}u_2u_3=0,$$

where either $p_{31}=0$ or $p_{20}=0$. If $p_{20}=0$, the equation of the involution conic is $u_1u_2=0$. The points are $u_2=0$ and $u_1=0$. The line $x_3=0$ counts for three common tangents, and the line $x_1=0$ is the fourth common tangent of the conics. If $p_{31}=0$, we have a similar situation in which u_1 and u_3 interchange their rôles. This discovery may be generalized to say that if $P=Q=0$, the involution conic degenerates

*For the case of 4-point contact, see below.

into two points, one of which is on the base conic, and the other on the tangent to the base conic at the first point. The conics may be said to osculate where the junction of the two points touches the conic. The tangent at this point counts for three common tangents, and the other tangent from the second point to the base conic is the fourth common tangent.

Let us now see whether $P=Q=0$ is a necessary condition that the conics osculate. We will again consider the cubic in k,

$$\Delta k^3+\Theta k^2+\Theta' k+\Delta'.$$

The condition that the conics osculate is

$$3\Delta/\Theta=\Theta/\Theta'=\Theta'/3\Delta'.$$

Therefore, if the base conic and the involution conic osculate,

$$-3\nabla^2/6\nabla P=6\nabla P/-9P^2=-9P^2/-12,$$

or

$$27\nabla^2 P^2=36\nabla^2 P^2,$$

$$-72\nabla P=81P^4,$$

and

$$36\nabla^2=-54\nabla P^3.$$

Hence, if the conics osculate, $\nabla=P=0$, and, therefore, $Q=0$. We have now proved the

THEOREM. *The necessary and sufficient condition that the conics have at least 3-point contact is $P=Q=0$. If the conics osculate the involution conic degenerates, but the converse is not necessarily true.*

9. *A study of the double points.* Before examining 4-point contact, let us study the double points of the involution to show the relation between them and the conics. Taking the cubics as in (3). the double points are given by the Jacobian of the two forms,

$$At^4+4Bt^3+6Ct^2+4Dt+E=0, \tag{14}$$

where $A=p_{01}/3$, $B=p_{02}/6$, $C=(p_{12}+3p_{03})/18$,

$$D=p_{13}/6, \quad \text{and} \quad E=p_{23}/3.$$

Since this is a quartic equation, it has two invariants which comprise its complete system of invariants. The discriminant Δ^* is a function of the two.

$$I_2:\quad AE-4BD+3C^2=P^2/12. \tag{15}$$

$$I_3:\quad ACE+2BCD-AD^2-B^2E-C^3=(Q-P^3/54)/4. \tag{16}$$

$$\Delta^*:\quad I_2^3-27I_3^2=-Q\nabla/16. \tag{17}$$

The necessary and sufficient condition that a binary quartic represent a set of four equi-anharmonic points is $I_2=0$. Therefore, if $I_2=0$, the double points of the involution are equi-anharmonic. From (15), we see that then $P=0$, which means the cubics are apolar and the conics are mutually apolar. Hence,

If the cubics are apolar (that is, the conics mutually apolar), the double points of the involution are equi-anharmonic, and conversely.

The necessary and sufficient condition that the double points be harmonic is that $I_3=0$ or $54Q-P^3=0$. When this happens the Jacobian factors into two apolar quadratics, *hence the double points are cut from the conic by a pair of conjugate polar lines.**

If $I_2=I_3=0$, the Jacobian has a triple point. If this relation is true, $P=Q=0$. This means that the conics osculate, and the double points reduce to two points, one taken three times.

The vanishing of the discriminant means that two double points are coincident. If $\Delta^*=0$, either $\nabla=0$ or $Q=0$. Now, if $\nabla=0$, the involution conic degenerates and has contact with the base conic, and if $Q=0$, the base conic and involution conic have contact. All these facts agree with our previous results.

10. *Four-point contact.* Let us now consider the conditions under which the conics have 4-point contact. First, we will find the necessary conditions by assuming they have contact at $t=0$. Since the Jacobian of the cubics will then take the form $t^4=0$, $p_{20}=p_{31}=p_{23}=0$, and $p_{12}=-3p_{03}$. Substituting these values, we find

$$P=2p_{03},\quad Q=4p^3_{03}/27,\quad \text{and}\quad \nabla=p^3_{30}.$$

If, then, the identity $\nabla=27Q-P^3$ is to hold, $p_{03}=p_{12}=0$. Hence $P=Q=\nabla=0$ and the base cubics are apolar, the involution contains a perfect cube and all sets have a common root. If p_{01} does not vanish, the equation of the involution conic becomes $u_3{}^2=0$. Since all other p_{ij} vanish the matrix of the cubics may be reduced to the canonical form

$$\begin{Vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix}.$$

Considering this matrix, we may summarize in the

Theorem. *If the two conics have four-point contact, the involution conic degenerates into that point repeated. All cubics in the involution have a common double root and the involution may be based on a cubic*

*Winger, *l. c.*, p. 251.

and its cubicovariant. Conversely, if all cubics in the involution have a common double root (point), the involtuion conic reduces to that point repeated and the involution may be based on a cubic and its cubicovariant.

11. *Summary of the cubic involution $I_{1,2}$.* Cubic involutions $I_{1,2}$ can also be classified by considering the $I_{1,2}$ cut from a (non-cuspidal) rational cubic curve by a pencil of lines in the plane. We shall summarize by correlating the types found by this method with the several cases that have just been discussed.

First, we have the general case, in which the point is such that four distinct tangents can be drawn to the cubic. The points of contact of the tangents are the double points of the involution and thus we have an involution with four sets containing a double point.

In the second case the point is on a flex tangent so that there are two distinct tangents to the cubic, whose contacts are two double points, and the other two tangents combine to form the flex tangent, whose contact is a triple point in the involution. Hence, we have an involution with one triple point and two double points. If we take as the two base cubics those whose matrix is

$$\begin{Vmatrix} a & 0 & 0 & 0 \\ 0 & 0 & c' & d' \end{Vmatrix},$$

we find that $Q=0$, and our two conics have contact at $t=0$.

We next have the case in which the involution has two sets with a triple point, which corresponds to our double-contact conics, and in in which the cubics take the form

$$\begin{Vmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & d' \end{Vmatrix}.$$

The center of the pencil of lines to the cubic is then the point of intersection of two flex tangents.

The case in which the involution conic degenerates into two points on the base conic at $t=0$ and $t=\infty$, and in which the matrix of the cubics is

$$\begin{Vmatrix} 0 & b & 0 & 0 \\ 0 & 0 & c' & 0 \end{Vmatrix},$$

may be represented by taking the center of the pencil on the double point of the cubic. Then each line cuts the cubic in the two points $t=0$ and $t=\infty$. Only two sets in the involution have a double point, one at $t=0$ and one at $t=\infty$.

If the center of the pencil is on the curve, all sets in the involution have a common point while the remaining pairs constitute a quad-

ratic $I_{1,1}$. Thus, $I_{1,2}=t(q_1+\lambda q_2)$. This corresponds to the case arising when f_1 and f_2 have a common point. That is $R=\nabla=0$, and the involution conic breaks up into two points.

Next, we have the case which corresponds to the osculating conics. From the center of the pencil can be drawn three coincident tangents and one other distinct tangent. This happens when the point is a flex point. The cubics may be represented by

$$\begin{vmatrix} a & 0 & 0 & 0 \\ 0 & 0 & c' & 0 \end{vmatrix}.$$

The last case corresponds to the 4-point contact conics, in which every set has a double point, and one has a triple point. This will be true if the center of the pencil is on the cusp of a cuspidal cubic, an exceptional case.

Analyzing Degrees of Freedom Into Comparisons When the "Classes" Do Not Contain the Same Number of Items

By William Dowell Baten
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Suppose that 20 items are divided into four groups G_1, G_2, G_3, G_4, each of which contains five items. Let t_1, t_2, t_3, and t_4 be respectively the totals of these groups. The analysis of the degrees of freedom is as follows:

Source of variation	*Degrees of freedom*
Total	19
Between group means	3
Within groups	16

The three degrees of freedom pertaining to groups, as is well known, can be separated into single degrees of freedom by the following comparisons:

$$t_1 - t_2$$

$$t_1 + t_2 - 2t_3$$

$$t_1 + t_2 + t_3 - 3t_4.$$

The sums of squares pertaining to these 3 comparisons are as follows:

$$\frac{(t_1 - t_2)^2}{5(1+1)}$$

$$\frac{(t_1 + t_2 - 2t_3)^2}{5(1+1+4)}$$

$$\frac{(t_1 + t_2 + t_3 - 3t_4)^2}{5(1+1+1+9)}.$$

Now consider that the groups do not contain the same number of items. The object of this short paper is to show how to analyze the degrees of freedom pertaining to groups into single degrees of freedom and then to derive expressions for the sums of squares pertaining to these degrees of freedom. Consider the case of two groups G_1 and G_2

containing respectively n_1 and n_2 items whose sums are respectively t_1 and t_2. The natural comparison to make is the difference between the group means, namely

$$\frac{t_1}{n_1} - \frac{t_2}{n_2} \quad \text{or} \quad \frac{n_2t_1 - n_1t_2}{n_1n_2}.$$

To find an expression for the sum of squares due to this comparison it is necessary to find h such that

$$\frac{(n_2t_1 - n_1t_2)^2}{n_1^2n_2^2h} = \frac{t_1^2}{n_1} + \frac{t_2^2}{n_2} - \frac{(t_1+t_2)^2}{n_1+n_2}.$$

From this equality the value of h is found to be

$$h = \frac{n_1+n_2}{n_1n_2};$$

hence the expression for the sum of squares due to this comparison is

$$\frac{(n_2t_1 - n_1t_2)^2}{n_1n_2(n_1+n_2)}. \tag{1}$$

In this expression the total of group G_1 has a weight equal to, n_2, the number of items in G_2 and the total of group G_2 has a weight equal to, n_1, the number of items in G_1. The denominator in (1) is equal to the product of the weights times the sum of the weights or $n_1n_2(n_1+n_2)$. If $n_1 = n_2$ (1) becomes

$$\frac{(t_1 - t_2)^2}{(1+1)n} \quad \text{as before.}$$

Consider the case of 3 groups G_1, G_2, G_3 with numbers of items and totals respectively n_1, n_2, n_3 and t_1, t_2, t_3. The 2 degrees of freedom pertaining to groups may be broken up into single degrees of freedom by the comparisons, if

$$N_j = \sum_1^j n_i \quad \text{and} \quad T_j = \sum_1^j t_i,$$

$$\frac{t_1}{n_1} - \frac{t_2}{n_2},$$

$$\frac{T_2}{N_2} - \frac{t_3}{n_3}.$$

or by the comparisons

$$\frac{n_2t_1-n_1t_2}{n_1n_2},$$

$$\frac{n_3T_2-N_2t_3}{n_3N_2}.$$

The expressions for evaluating the sums of squares due to these comparisons are

$$\frac{(n_2t_1-n_1t_2)^2}{n_1n_2N_2},$$

$$\frac{(n_3T_2-N_2t_3)^2}{n_3N_2N_3}.$$

The second expression in the above can be found by equating two quadratic expressions in t_1, t_2, and t_3. The second expression can also be found by using the same rule for finding the first expression by considering that the second comparison is the difference of two averages namely the average of the items in groups G_1 and G_2 and the average of the third group. The denominator in the expression for the sum of squares due to this comparison is equal to the product of the weights times the sum of the weights or $n_3\ N_2\ N_3$.

Consider the general case of s groups with the number of items and totals respectively $n_1, n_2, \cdots, n_s$ and $t_1, t_2, \cdots, t_s$. Reasoning as before, one set of comparisons is (without denominators)

$$
\begin{array}{l}
n_2T_1-N_1t_2 \\
n_3T_2-N_2t_3, \\
n_4T_3-N_3t_4 \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
n_kT_{k-1}-N_{k-1}t_k \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
n_sT_{s-1}-N_{s-1}t_s.
\end{array}
$$

The sums of squares due to these comparisons are respectively:

$$r_1 \quad : \quad (n_2t_1-n_1t_2)^2/n_1n_2N_2$$

$$r_2 \quad : \quad (n_3T_2 - N_2t_3)^2/n_3N_2N_3,$$

$$\cdots\cdots\cdots\cdots$$

$$r_{k-1} \quad : \quad (n_kT_{k-1} - N_{k-1}t_k)^2/n_kN_{k-1}N_k,$$

$$\cdots\cdots\cdots\cdots$$

$$r_{s-1} \quad : \quad (n_sT_{s-1} - N_{s-1}t_s)^2/n_sN_{s-1}N_s.$$

R. A. Fisher (1) proved the following theorem. "If $x_1, x_2, \cdots, x_s$ are distributed normally and independently with unit standard deviation and if

$$w_1 = p_{11}x_1 + p_{12}x_2 + \cdots + p_{1s}x_s,$$

$$w_2 = p_{21}x_1 + p_{22}x_2 + \cdots + p_{2s}x_s,$$

$$\cdots\cdots\cdots\cdots$$

$$w_s = p_{s1}x_1 + p_{s2}x_2 + \cdots + p_{ss}x_s,$$

then w_i will be distributed normally and independently with unit standard deviation, provided for all values of i

$$p_{i1}^2 + p_{i2}^2 + \cdots + p_{is}^2 = 1$$

and for all values of i and j where $i \neq j$

$$p_{i1}p_{j1} + p_{i2}p_{j2} + \cdots + p_{is}p_{js} = 0''.$$

This theorem will be true if the standard deviation of each x is σ. He also showed that if $w_1, w_2, \cdots, w_h$ are any linear functions of the x's, fulfilling these conditions and $h < s$, then

$$\sum_1^s x^2 - \sum_1^h w_i^2$$

can always be expressed in the form

$$\sum_{i=h+1}^{s} w_i^2,$$

this being wholly independent of $w_1, w_2, \cdots, w_h$.

Consider the following linear functions of $t_i/\sqrt{n_i}$, where t_i is the sum of the variates in the ith group which are normally and independently distributed, each with standard deviation equal to σ.

$$w_1 = \frac{\sqrt{n_2n_1}}{\sqrt{N_2 \cdot N_1}} \cdot \frac{t_1}{\sqrt{n_1}} - \frac{\sqrt{n_1}}{\sqrt{N_2}} \cdot \frac{t_2}{\sqrt{n_2}},$$

$$w_2 = \frac{\sqrt{n_3 n_1}}{\sqrt{N_2 \cdot N_3}} \cdot \frac{t_1}{\sqrt{n_1}} + \frac{\sqrt{n_3 n_2}}{\sqrt{N_2 \cdot N_3}} \cdot \frac{t_2}{\sqrt{n_2}} - \frac{\sqrt{N_2}}{\sqrt{N_3}} \cdot \frac{t_3}{\sqrt{n_3}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$w_{k-1} = \frac{1}{\sqrt{N_{k-1} N_k}} \sum_{i=1}^{k} \sqrt{n_k n_i} \frac{t_i}{\sqrt{n_i}} - \frac{\sqrt{N_{k-1}}}{\sqrt{N_k}} \cdot \frac{t_k}{\sqrt{n_k}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$w_s = \frac{1}{\sqrt{N_s}} \sum \sqrt{n_i} \frac{t_i}{\sqrt{n_i}},$$

where $k = 1, 2, \cdots, s-1$. From the assumption concerning the variates in each group, the standard deviation of each $t_i/\sqrt{n_i}$ is σ since $t_i/\sqrt{n_i} = \sqrt{n_i}\,\bar{x}_i$; and since $t_i/\sqrt{n_i}$ and $t_j/\sqrt{n_j}$ are independent. The standard deviation of each w_i, $i = 1, 2, \cdots, s$ is also equal to σ.

The sum of the squares of the coefficients of each of the above linear expressions is unity; for consider the sum of the squares of the coefficients of w_{k-1}. The sum is

$$\frac{n_k N_{k-1}}{N_{k-1} N_k} + \frac{N_{k-1}}{N_k} = \frac{1}{N_k}(n_k + N_{k-1}) = \frac{N_k}{N_k} = 1.$$

The sum of the squares of the coefficients in w_s is unity as can be seen immediately. The sum of the products of the corresponding coefficients of w_{k-1} and w_{q-1}, $k<q$ is

$$\frac{\sqrt{n_k n_q} N_{k-1}}{\sqrt{N_{k-1} N_k N_{q-1} N_q}} - \frac{\sqrt{N_{k-1} n_k n_q}}{\sqrt{N_k N_{q-1} N_q}}$$

or

$$\frac{\sqrt{n_k n_q}}{\sqrt{N_k N_{q-1} N_q}} \left[\frac{N_{k-1} - N_{k-1}}{\sqrt{N_{k-1}}} \right] = 0.$$

In a similar manner the sum of the products of the corresponding coefficients of w_{k-1} and w_s can be shown to be equal to zero.

According to the above theorem

$$\sum_1^{s-1} w_i^2 = \sum_1^{s} (t_i/\sqrt{n_i})^2 - w_s^2 \quad \text{or}$$

$$(2) \qquad \sum_{i=1}^{s-1} w_i^2 = \sum_{1}^{s} t_i^2/n_i - \frac{(\Sigma t_i)^2}{N_s}.$$

The right-hand member of the above equation is the sum of squares due to variation among the group means, while the left-hand member is the sum of the sums of squares pertaining to the single degrees of freedom. The w_i, $i=1, 2, \cdots, s-1$, are independent. The square of w_i has one degree of freedom since the square of a linear expression, under the conditions considered here, has one degree of freedom. Since

$$w_i^2 = r_i, \; i=1, 2, \cdots, s-1,$$

it follows that the sum of the r's is equal to the sum of squares *between* group means.

A second method of proving identity (2), is to show that the coefficients of t_i^2 in the left-hand member is equal to $(1/n_i - 1/N_s)$, its coefficient in the right-hand member and to show that the coefficient of $2t_it_j$, $i<j$, in the former member is equal to $-1/N_s$, its coefficient in the latter. The quantity t_i^2 appears for the first in w_{i-1}^2 and thereafter in all w_j^2, for $j>i-1$. This coefficient of t_i^2 is

$$(3) \qquad \frac{N_{i-1}}{N_in_i} + \frac{n_{i+1}}{N_iN_{i+1}} + \frac{n_{i+2}}{N_{i+1}N_{i+2}} + \frac{n_{i+3}}{N_{i+2}N_{i+3}} + \cdots \frac{n_s}{N_{s-1}N_s}.$$

The sum of the first two terms in (3) is

$$\left(\frac{1}{n_i} - \frac{1}{N_{i+1}}\right);$$

the sum of the first three terms is

$$\left(\frac{1}{n_i} - \frac{1}{N_{i+2}}\right);$$

the sum of the first four terms is

$$\left(\frac{1}{n_i} - \frac{1}{N_{i+3}}\right), \text{ etc.};$$

hence the sum of all the terms in (3) is

$$\left(\frac{1}{n_i} - \frac{1}{N_s}\right).$$

The coefficient of $2t_it_j$ in the left-hand member of (2) can be found. The quantities t_i and t_j, $j>i$ appear for the first time in w_j^2 and thereafter in all w_k^2, $k>j$. This coefficient of $2t_it_j$ is

$$(4) \qquad -\frac{1}{N_j} + \frac{n_{j+1}}{N_jN_{j+1}} + \frac{n_{j+2}}{N_{j+1}N_{j+2}} + \frac{n_{j+3}}{N_{j+2}N_{j+3}} + \cdots + \frac{n_s}{N_{s-1}N_s}.$$

The sum of the first two terms of (4) is equal to $-1/N_{j+1}$; the sum of the first 3 terms is $-1/N_{j+2}$; the sum of the first 4 terms is $-1/N_{j+3}$, etc.; the sum of the terms in (4) is $-1/N_s$. The coefficient of this term in the right-hand member of (2) is also equal to $-1/N$. Hence (2) is true. Since the degree of freedom for each w_i^2 is 1, the linear expressions w_i are independent by Cochran's theroem (2).

Irwin did something like this by using unequal weights. (3).

Applications

Table 1 contains the yields of tomatoes grown in three fertilizers. There are 6 values for each of fertilizers 1 and 3 and 5 for fertilizer 2. Table 2 contains the analysis of variance pertaining to these yields.

TABLE 1

DATA CONCERNING YIELDS OF U. S. NO. 1 AND NO. 2 COMBINED FOR 1937 TOMATO FERTILIZER PLOTS. (pounds)

	Fertilizers 1	2	3
	161	190	145
	199	224	163
	133	179	181
	203	182	137
	158	203	129
	190		163
Ave.	174	196	153

TABLE 2

ANALYSIS OF VARIANCE OF TOMATO YIELDS.

Source of variation	*D. F.*	*Sum of Squares*	*Mean Square*
TOTAL	16	11,714.1	
FERTILIZER COMPARISONS			
$n_3t_1-n_1t_3$	1	1,386.7	1,386.7
$n_2t_1+n_2t_3-(n_1+n_3)t_2$	1	3,693.6	3,693.6*
Within—error	14	6,633.8	473.9

*Significant at the 5% level.

The sums of squares due to the comparisons pertaining to fertilizers in Table 2 are as follows:

$$(n_3t_1-n_1t_3)^2/n_1n_3(n_1+n_3)=(1044-915)^2/12=1{,}386.75$$

$$[n_2t_1+n_2t_3-(n_1+n_3)t_2]^2/n_2(n_1+n_3)N$$
$$=(5x1959-12x978)^2/5x12x17=1941^2/1020=3,693.61.$$

Adding these one gets the sum of squares due to fertilizers equal to 5080.4.

Another use of these comparisons and the expressions for the sum of squares due to them is made when missing plots arise in randomized block layouts or layouts with more than one classification.

LITERATURE

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The Magnilong Near-Laguerre Transformations*

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1. *Introduction.* In this paper, we shall begin the study of the group of line transformations of the complex plane with reference to the maximum number of circles preserved.[1] To do this simply, it is found convenient to introduce the following preliminary definitions and observations. The group of line transformations may be classified into the following *three* distinct types: (I). The group of *magnilong* transformations. Any correspondence T of this group magnifies by a constant $\gamma \neq 0$ the distance between the points of contact on the common tangent line of any two curves. If $\gamma = 1$, the resulting correspondence is called an *equilong* transformation. From this, it follows that our group of magnilong transformations may be factored into the product of the group of equilong transformations by the group of ordinary magnifications. Obviously any magnilong correspondence preserves the ∞^1 parallel pencils of lines. (II). The set of *affinilong* transformations. Any correspondence T of this set preserves every parallel pencil of lines, but it is *not* a magnilong transformation. A nonmagnilong correspondence T is an affinilong transformation if and only if the ratio, into which one of the three points of contact on the common tangent line of any three curves divides the segment determined by the other two points, is preserved. (III). The set of general transformations. Any correspondence T of this set is *not* of the types (I) and (II). Thus T is a general transformation if and only if it does *not* preserve every parallel pencil of lines.

In a previous paper,[2] we discussed the point transformations of the plane with reference to the maximum number of vertical parabolas preserved. This discussion is isomorphic to that of the line transformations with reference to the maximum number of circles preserved. We shall now state some of the results of the preceding paper from the latter point of view. A general line transformation of the complex plane converts at most $2\infty^2$ circles into circles.[3] A nongeneral line transformation always carries ∞^1 circles into circles. These are the

*Presented to the American Mathematical Society, February, 1940.

[1] By a circle, we mean in line geometry either (a) a proper circle considered as the envelope of its tangent lines, or (b) a pencil of lines with finite vertex, or (c) a parallel pencil of lines.

∞^1 parallel pencils of lines which become parallel pencils of lines. An affinilong line transformation carries at most ∞^2 circles into circles. A magnilong line transformation preserves at most ∞^1 circles (besides the ∞^1 parallel pencils of lines). A correspondence is called a Laguerre transformation if and only if it carries the entire family of ∞^3 circles into circles.

From the preceding facts, we obtain the following three minimal characterizations of the seven-parameter group G_7 of Laguerre transformations: (1). If $3\infty^2$ circles are carried into circles under a line transformation, the same is true of all circles, and the line transformation is therefore a Laguerre transformation. (2). If $2\infty^2$ circles are carried into circles under a nongeneral line transformation, the same is true of all circles, and the nongeneral line transformation is therefore a Laguerre transformation. (3). Any magnilong transformation which converts $2\infty^1$ circles (besides the ∞^1 parallel pencils of lines) into circles is a Laguerre transformation. This idea of a minimal characterization of a given group of transformations was first developed by Kasner.[4]

In this paper, we shall determine the set of all *magnilong near-Laguerre transformations* (in analogy with the term near-collineation as defined by Kasner). That is, we shall obtain the set of all magnilong transformations which convert *exactly* ∞^1 circles into circles (besides the ∞^1 parallel pencils of lines). *Any magnilong near-Laguerre transformation is of the form* L_2TL_1 *where* L_1 *and* L_2 *are Laguerre transformations and* T *is any of the three transformations* e^z, $\log z$, z^n *where* z *represents the dual complex number* $z=x+jy$, (x, y *complex equilong line coordinates and* $j^2=0$). The family is a pencil of circles in line geometry (that is, the set of ∞^1 circles tangent to two fixed lines).

To study the line geometry of the plane, we usually define a lineal element E by the *hessian coordinates* (u, v, w) where v is the length of the perpendicular from the origin, u is the angle between the initial line and the perpendicular, and $w=dv/du$ is the distance between the

[2] De Cicco, *The analogue of the Mœbius group of circular transformations in the Kasner plane*, Bulletin of the American Mathematical Society, Vol. 45, pp. 936-943, (1939). Also see Kasner and De Cicco, *Characterization of the Mœbius group of circular transformations*, Proceedings of the National Academy of Sciences, Vol. 25 (1939), pp. 209-213. Kasner and De Cicco, *The conformal near-Moebius transformations*, Bulletin of the American Mathematical Society, Vol. 46, pp. 784-793 (1940).

[3] In the previous paper, we derived these results for the line transformations o the *real* plane. But these theorems may be extended to the complex plane without any difficulty. Note that a given family F of geometric configurations in the complex plane is said to possess ∞^n configurations if each of these is determined uniquely by n complex contants.

[4] Kasner, *The characterization of collineations*, Bulletin of the American Mathematical Society, Vol. 9 (1903), pp. 545-546.

foot of the perpendicular and the point of the element E. But for our purposes, we shall find it more convenient to define a lineal element E by the *equilong coordinates* $(x, y, p=dy/dx)$. The hessian and the equilong coordinates are connected by the relations

$$x=\tan u/2, \quad y=\tfrac{1}{2}v \sec^2 u/2, \quad p=w+v\tan u/2. \tag{1}$$

The inverse of this correspondence is

$$u=2\arc\tan x, \qquad v=\frac{2y}{1+x^2}, \quad w=p-\frac{2xy}{1+x^2}. \tag{2}$$

2. *The form of the differential equation of the invariant family* (∞^1) *of circles.* In hessian coordinates, the equation of any circle C is

$$v=a\cos u+b\sin u+r, \tag{3}$$

where (a, b) are the cartesian coordinates of the center and r is the radius of C. Replacing u and v by the values given in (2), we find that the ∞^3 circles of the plane are represented in equilong coordinates by the ∞^3 vertical paravolas

$$a_0x^2+a_1x+a_2y+a_3=0, \tag{4}$$

where at least one of the coefficients of the unknowns is not zero. If $a_2\neq 0$, then (4) represents a circle whose center (a, b) and radius r

$$a=\frac{a_0-a_3}{a_2}, \qquad b=-\frac{a_1}{a_2}, \quad r=\frac{-a_0-a_3}{a_2}. \tag{5}$$

Otherwise if $a_2=0$, the circle (4) represents one or two parallel pencils of lines.

From (4), it is found that in equilong coordinates the differential equation of the entire family of ∞^3 circles is

$$p''=0 \qquad (p=dy/dx). \tag{6}$$

It is noted that in equilong coordinates, the Laguerre group G_7 of circular line transformations is

$$X=\frac{ax+b}{cx+d}, \qquad Y=\frac{ey+fx^2+gx+h}{(cx+d)^2}, \tag{7}$$

where a, b, c, d, e, f, g, h are constants and $ad-bc\neq 0$. If in these equations, we take e to be the quantity $ad-bc$, the resulting six-parameter group may be written in the compact form

$$Z=\frac{\alpha z+\beta}{\gamma z+\delta}, \tag{8}$$

where each of the quantities z, Z, α, β, γ, δ represents a *dual complex number* of the form $a+jb$ where $j^2=0$ and a and b are complex numbers. Then any Laguerre transformation is the product of a linear fractional correspondence of the above form followed by the magnification $X=x$, $Y=ey$.[5]

In equilong coordinates, any magnilong transformation is given

$$X=\Phi(x), \qquad Y=\gamma\Phi_x y+\Psi(x), \tag{9}$$

where $\gamma\neq 0$ is constant and $\Phi_x\neq 0$. Upon extending this magnilong transformation three times, we obtain

$$P=\gamma p+\frac{\gamma\Phi_{xx}}{\Phi_x}y+\frac{\Psi_x}{\Phi_x},$$

$$P'=\frac{\gamma}{\Phi_x}p'+\frac{\gamma\Phi_{xx}}{\Phi_x^{\,2}}p+\frac{\gamma}{\Phi_x}\left(\frac{d}{dx}\,\frac{\Phi_{xx}}{\Phi_x}\right)y+\frac{1}{\Phi_x}\left(\frac{d}{dx}\,\frac{\Psi_x}{\Phi_x}\right), \tag{10}$$

$$P''=\frac{\gamma}{\Phi_x^{\,2}}p''+\frac{\gamma(2\Phi_x\Phi_{xxx}-3\Phi^2_{xx})}{\Phi_x^{\,4}}p+\frac{\gamma y}{\Phi_x}\,\frac{d}{dx}\left(\frac{1}{\Phi_x}\,\frac{d}{dx}\,\frac{\Phi_{xx}}{\Phi_x}\right)+\frac{1}{\Phi_x}\,\frac{d}{dx}\left(\frac{1}{\Phi_x}\,\frac{d}{dx}\,\frac{\Psi_x}{\Phi_x}\right).$$

For those circles which become circles under our magnilong transformation, we know that the differential condition (6) must be preserved. Upon applying these conditions to our magnilong transformation, we obtain the

THEOREM 1. *The only possible circles (besides the* ∞^1 *parallel pencils of lines) which become circles under the magnilong transformation* (9), *not of the Laguerre type, are the* ∞^1 *circles whose differential equation is of the form*

$$\frac{\gamma(2\Phi_x\Phi_{xxx}-3\Phi^2_{xx})}{\Phi_x^{\,3}}p=-\gamma y\frac{d}{dx}\left(\frac{1}{\Phi_x}\,\frac{d}{dx}\,\frac{\Phi_{xx}}{\Phi_x}\right) \tag{11}$$

[5] The Laguerre group is usually defined to be that of the linear fractional correspondences of the form (8). But in our work, we shall take it to mean the entire group of line transformations which preserve the ∞^3 circles of the plane.

$$-\frac{d}{dx}\left(\frac{1}{\Phi_x}\frac{d}{dx}\frac{\Psi_x}{\Phi_x}\right).$$

It therefore remains to find the shape of the ∞^1 circles whose differential equation in equilong coordinates is in this form (11).[6]

3. *The* ∞^1 *circles whose differential equation is of the form* (11). The family of circles is given by a differential equation of the form

$$p=\lambda(x)+y\mu(x). \tag{12}$$

We shall find all families of circles whose differential equation is of this form. By means of the Laguerre transformations, we shall reduce our results to canonical forms.

The complete solution of this differential equation (12) must be of the form

$$y=\alpha(x)+k\beta(x), \qquad \beta\neq 0, \tag{13}$$

where k is an arbitrary constant. This equation will represent ∞^1 circles if and only if it is of the form

$$y=(\alpha_0+\alpha_1x+\alpha_2x^2)+k(\beta_0+\beta_1x+\beta_2x^2), \tag{14}$$

where $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2$, are fixed constants. This represents a pencil of circles in line geometry. That is, the above represents the ∞^1 circles which are tangent to two fixed lines (which may be coincident).

Upon applying appropriate Laguerre transformations to this equation (14), we obtain the following

THEOREM 2. *The only* ∞^1 *circles whose differential equation is of the form* (11) *or* (12) *are those of a pencil of circles in line geometry. That is, the circles must be tangent to two fixed lines* (*which may be coincident*). *Under the group of Laguerre transformations, all the pencils of circles may be classified into two distinct types:* (1). *The parabolic pencils of circles, and* (2). *The nonparabolic pencils of circles.*

The canonical form of any parabolic pencil of circles is

$$y=\text{const}, \quad \text{or} \quad p=0. \tag{A}$$

This represents the ∞^1 circles which are tangent to each other at the origin in the negative direction of the perpendicular to the initial line. The canonical form of any nonparabolic pencil of circles is

$$y=\text{const. } x, \quad \text{or} \quad p=y/x. \tag{B}$$

[6] Note that the coefficient of p in (11) cannot be zero. For otherwise the only circles which are preserved under our magnilong transformation (9), not of the Laguerre type, are the ∞^1 parallel pencils of lines.

This represents the ∞^1 points (the finite vertices of the ∞^1 pencils of lines) on the perpendicular to the initial line at the origin.

4. *The magnilong near-Laguerre transformations.* Now we proceed to find the magnilong near-Laguerre transformations. First it is obvious that *the inverse N^{-1} of any near-Laguerre transformation N is also a magnilong near-Laguerre transformation.* For since N is a one-to-one correspondence which carries ∞^1 circles into circles (excluding the ∞^1 parallel pencils of lines), it follows that the transformed circles must be ∞^1 in number. Hence the inverse N^{-1} carries ∞^1 circles (excluding the ∞^1 parallel pencils of lines) into circles, and therefore it must be a magnilong near-Laguerre transformation.

Let N be any magnilong near-Laguerre transformation. There exist exactly ∞^1 circles (excluding the ∞^1 parallel pencils of lines) which are preserved by N. Denote these circles by γ and their transformed circles under N by Γ. That is, $N(\gamma)=\Gamma$ and $\gamma=N^{-1}(\Gamma)$. By Theorems 1 and 2, we find that *both γ and Γ are each pencils of circles.*

Let γ_0 denote the canonical forms (A) or (B) of a linear pencil of circles. Let L_1 be any Laguerre transformation which carries the pencil of circles γ into the canonical form γ_0, and let L_2 be any Laguerre transformation which converts the canonical form γ_0 into the pencil of circles Γ. Then the line transformation $T=L_2^{-1}NL_1^{-1}$ preserves the canonical form γ_0 of the pencil of circles. Hence *any magnilong near-Laguerre transformation N is of the form L_2TL_1 where L_1 and L_2 are Laguerre transformations and T is any magnilong near-Laguerre transformation which preserves the canonical form γ_0 of a pencil of circles.*

Next we shall find all the transformations T which preserve the canonical form γ of a pencil of circles. Obviously T converts either (I). (A) into (A), or (II). (A) into (B) or (III). (B) into (A), or (IV). (B) into (B)

(I). Let T convert (A) into (A). The differential equation $p=0$ must be preserved. For this to be so, we find that T must be of the form $X=ax+b$, $Y=cy+d$, where $a\neq 0$, b, $c\neq 0$, d are constants. This of course is a simple case of the Laguerre transformations. Hence *here are no magnilong near-Laguerre transformations which preserve a parabolic pencil of circles.*

(II). Let T convert (A) into (B). The differential equation $p=0$ is converted into $P=Y/X$. For this to be so, we find that T must be of the form $X=be^{ax}$, $Y=(cy+d)e^{ax}$, where $a\neq 0$, $b\neq 0$, $c\neq 0$, d are constants. Thus, *any magnilong near-Laguerre transformation N which carries a parabolic pencil of circles into a nonparabolic pencil of circles must be of the form L_2TL_1 where L_1 and L_2 are Laguerre transforma-*

tions and T is the transformation $Z=e^z$ where z, Z represent dual complex numbers.

(III). Let T convert (B) into (A). The differential equation $p=y/x$ is converted into $P=0$. Thus T must be of the form $Y=a \log x+b$, $Y=cy/x+d$, where $a\neq 0$, b, $c\neq 0$, d are constants. Therefore *any magnilong near-Laguerre transformation N which carries a nonparabolic pencil of circles into a parabolic pencil of circles must be of the form L_2TL_1 where L_1 and L_2 are Laguerre transformations and T is the transformation $Z=\log z$*

(IV). Let T convert (B) into (B). Then the differential equation $p=y/x$ must be preserved. Hence T must be of the form $X=ax^n$, $Y=bx^{n-1}y+cx^n$, where $n\neq 0$, $a\neq 0$, $b\neq 0$, c are constants. Thus *any magnilong near-Laguerre transformation N which carries a nonparabolic pencil of circles into a nonparabolic pencil must be of the form $L\,TL_1$ where L_1 and L_2 are Laguerre transformations and T is the transformation $Z=z^n$(n complex constant).*

We obtain from the preceding results the

Theorem 3. *Any magnilong near-Laguerre transformation of the complex plane is of the form L_2TL_1 where L_1 and L_2 are Laguerre transformations and T is any one of the three transormation* (1) $Z=e^z$; (2) $Z=\log z$; *and* (3) $Z=z^n$ *wh re n is a complex nonzero constant and where z represents the dual complex number $x+jy$(x, y are complex equilong line coordinates and $j^2=0$.*

In this paper, we have given all the magnilong near-Laguerre transformations, that is, al the magnilong transformations which preserves exactly ∞^1 circles (besides the ∞^1 parallel pencils of lines which are preserved). In later papers, we shall give (1) the set of all *affinilong near-Laguerre transformations*, that is, al the affinilong transformations which preserve *exactly* ∞^2 circles; and (2) the set of all *general near-Laguerre transformations*, that is, all the general line transformations which preserve *exactly* $2\infty^2$ circles.

Humanism and History of Mathematics

Edited by
G. Waldo Dunnington and A. W. Richeson

The Origin and Development of Tables of Weight, Length and Time

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A study of the origin and development of arithmetical tables which we use every day is of considerable importance to one who is interested in the History of Mathematics. We use them as a standard and seldom stop to consider from whence they came. They have been passed on from generation to generation, and for the most part, we find very little change in their comparative values.

A great number of our length measures come from parts of the body. For instance, the *ell* was the length from the elbow to the middle finger tip; the *yard*, the distance from the arm pit to the longest finger tip, and the *span*, the distance between the extended thumb tip and the longest finger tip. A *pace* was two steps, and a thousand paces a *mile*. The *league* was supposed to have been the distance seen by the naked eye on a level plane. And there are many others of similar origin, such as the *furlong, handsbreadth, foot,* and the *fathom.*

Probably the oldest use of a measure of length, the *foot,* was made by the Hindus. The *inch* was equal to three *barley corns*† laid lengthwise, or the length of eight *barley corns* laid side by side. An *inch* was also considered to be the length to the first joint of the middle finger. The Arabs received this method of measure from the Hindus.

The Egyptians had a system based on the *cubit,* which was the distance from the elbow to the longest finger tip. A *cubit* was equal to two *spans,* a *span* being divided into twelve fingers or digits. But this system was almost as inaccurate as others of those early days because of the difference in the length of the *cubit.*

From the Arabs, the Romans received some of their tables. The Roman *uncia* was the distance from the joint to the tip of the longest

*Assisted by Barbara Jean Blake, Senior in Albion College.
†Three barley corns are practically equivalent to one inch.

finger, the same as the Hindus' inch. Twelve of the Roman *unciæ* were supposed to be the length of a man's foot, thus making the *uncia* not quite an inch, as the Roman foot averages about 11.62 inches. There was some variance in the length of the foot, but even so a standard length was set and used throughout the Roman empire, and all of the measurements except the rod were dependent upon it. When the empire broke up, each nation adopted its own measure of the foot, and in 1500 we could find almost as many different lengths for it as there were countries. Their *pace* was equal to five feet and a mile a thousand *paces*, thus the Roman mile was five thousand feet.

When William the Conquerer became ruler of Eng and, he set up the following tables:

3 barley corns	1 inch
12 inches	1 foot
3 feet	1 yard
5½ yards	1 perch

At the time of Edward III, the length of the monarch's arm was a standard measure. Thus we observe that the standard of measure was changed with each ruler. A little later, the term *yard* was substituted for *perch*, and the yard became the usual measure of cloth. However, there became a tendency to measure the thumb's breadth as a part of the yard measure, making a yard anywhere from 36 to 37 inches, as well as being an inaccurate measurement. To do away with this inaccuracy, Queen Anne directed the standard yard to be 37 inches, but this still did not account for the error. Even today, unless machine measured, we are likely to find a difference as great as a thumb's breadth in a yard of cloth.

It was about this same time that the *acre* measurement was brought into use. At first an *acre* was considered to be the amount of land that a yoke of oxen could plow in one day. Naturally, this was very inconstant on account of the differences of land, speed of the oxen, etc. Henry VII set up the *rod* measurement as 16½ feet, and an *acre* as a strip of land 40x4 rods. A little later the *acre* was defined as 16x10 rods. The shape of the tract was as important as the amount of land. At this same time the mile was establ shed as 320 rods and a square mile equal to 640 acres. With this measurement of the mile in terms of rods, the mile is 5,280 feet, the standard used today.

These tables were used for some time. About the fourteenth century the French, in an endeavor to establish a nonvariable system, set up as a length, the length of a second's pendulun at forty-five degrees latitude; this length, named a *meter* was divided into 39.37 parts, 36 of which made up a yard. Each yard was d vided into three

parts, and each part, the foot, was divided into twelve parts, the standard inch. While the metric system was established earlier than the above, it was not adopted and made legal until 1840.

Two systems of weights have been developed, troy and avoir du pois. Both are used to some extent today, but the avoir du pois is more common. Troy weight is used for weighing jewels and precious stones. The origin of these two terms is very interesting. At that time, there were two relative weights for what is now known as the pound. England used the larger unit, the *libra*, and developed the merchant's pound, later named avoir du pois, meaning *pound of goods* in French. At a coin mint in Troy, England, the smaller unit of weight was used to weigh silver and other coin metals. This became known as the troy pound.

The *gerah*, the most ancient of the weight systems, originated with the Egyptians. In 2000 B. C. the Egyptians took a container, each dimension being equal to one cubit, and filled it with water. The weight of the water was divided into sixty equal parts, and each part was subdivided into 1000 parts. Each of these subdivisions was called a *gerah*. Twenty *gerah* made up a *shekel*. The *gerah* was also equal to twelve grains according to the Hindus. The grain was the fundamental unit, equal to the weight of a kernel of barley. The use of the grain for measuring weight is the oldest known base. The grain used was the seed of a creeping plant, the *rati*. The Hindus also used the *carat*, which was a bean, the fruit of an Abyssinian tree. It is now used in determining fineness rather than a definite measure of weight. The seeds, the *rati* and *carat*, were used because the seeds were always about the same weight. The grain was a popular measure of weight for several centuries. Even as late as the thirteenth century an English penny had to weigh thirty-two wheat corns taken from the center of the ear; twenty pennies an ounce; and twelve ounces a pound.

The Romans used a pound of 5204 grains which was divided into twelve parts, named *unciæ*, twelve of which made a pound in troy weight. It is from this term that our word *ounce* comes. For measuring small quantities they used the *denarius*, the weight of the silver penny. This term was also used on Roman coins signifying the coins of lesser value. From the *denarius* came the word *drachme*, used by physicians in measuring medicines. The *drachme*, though used today, was an old term even in the first century A. D. A little later the Romans desired to use a more exact measurement of weight. They filled a container, each dimension equal to one foot, with rain water. The total weight of the water was called a *talent*. The *talent* was divided into a thousand parts, twelve of which made one *libra*. One part was called a *uncia*, later an ounce.

For the purpose of weighing, the eastern nations made early use of the two-pan balance. This is still used in some places. For weights they used pieces of silver bear ng an official mark. A little later pieces of silver which has the picture of the ruler imprinted on them could be used either as a coin or as a standard weight. This explains how the *shekel* became the name of a coin. The piece of silver weighing twenty *gerahs* weighed one *shekel* and in 135 B. C. became the name of the coin of that weight. The Romans found that bronze was more useful and more durable than silver, so many of the standard weights were bronze.

When the Teutonic tribes invaded the Roman Empire, they introduced a weight unit larger than the *libra*, which they called a *pond*, or *pund*. It was made of iron and was used with a weighing device known as the steelyard, a long notched steel bar. This scale required only one weight, whereas the pan balance required a different weight for each different unit. The steelyard balance scale came to be used almost entirely for ordinary purposes, while the two-pan balance was used for accurate and careful weighing. The *libra* and *pond* were so difficult to distinguish between that in a short time they became equal. It is from the *libra* that we get our abbreviation for pound, lb. This is not accurate as the *libra* was not equal to the pound of today, but is an approximation, the two being relative weights.

It became necessary to have a measure of capacity as well as a measure of mass. In the second century Galen discovered a device for measuring liquid. It had the shape of a graduated horn called the *libra*. The *libra* was divided into twelve parts each called a *uncia*, the same as the mass weight. In general liquids were measured by the *libra*, and solids by the pound.

This constituted the weight measurements for several centuries. In the fifteenth century, an elaborate table was set up by Henry VII. One *sterling* should be equal to 32 grains of wheat. There were 20 *sterlings* per *ounce*, twelve ounces per *pound*, eight pounds per *gallon*, eight gallons per *bushel*. From this table has come the old saying "a pint equals a pound". This was later called the troy table. The avoir du pois was set up on the basis of 7000 grains per pound, while the weight of the troy pound was 5760. Hence both of the tables are based on the grain.

In the eighteenth century the French united with the English in attempting to find a unit of weight which would be more constant than the grain. The British government finally succeeded early in the nineteenth century in finding an invariable measure of length and from this was determined an invariable measure of weight. A one inch cube was filled with distilled water which was divided into 352,422

parts; 7000 of these parts were equal to the British *avoir du pois pound*. Likewise the British *gallon* was equal to ten of these pounds; thus the liquid measure was established.

However, it was not until the latter part of the nineteenth century that the divisions of the pound into *ounce* and *dram*, and of the gallon into *quart* and *pint*, were established. From this time on the Troy weight seems to have decreased in use, until today it is used only for weighing gold, silver, platinum, or other precious metals and stones.

This constitutes the table of weights for the most part. There are, however, some multiples of the pound, the *hundred-weight* and the *ton*. A hundredweight is taken to weigh 112 pounds. This weight comes from weighing 100 pounds, each pound separately. The final weight obviously would be more than a hundred pounds, on account of the inaccuracy of weighing such a large number of small articles. The table used today is:

Troy

24 grains	1 pennyweight
20 pennyweights	1 ounce
12 ounces	1 pound
5760 grains	1 pound

Avoir du pois

16 drams	1 ounce
16 ounces	1 pound
112 pounds	1 hundredweight
20 hundredweight	1 ton
50 pounds	1 half cental

This shows the ton equal to 2240 pounds. and is the standard measure. In the United States, however, the ton is often used as 2000 pounds, being 20 hundredweights of 100 pounds each.

Let us now look at some of the more recent developments and uses of these tables, more specifically, in the United States. Of course most of our ideas of measurement came from England. As the people came over, each brought his own methods of measuring, and made new ones to fit his needs.

When the constitution was written, a provision was made giving Congress the right to set up the standard tables of weights and measures. But at this time, there were many different systems. As no particular conflict arose on account of these differences, Congress made little attempt to establish uniform tables. The troy system of weights was recognized and adopted for use in coinage in 1828.

In the early part of the nineteenth century they obtained from England a brass bar, eighty-two inches long. It was divided into inches and tenths of an inch. After a long deliberation, it was decided that when the temperature of the rod was sixty-two degrees, the distance from the twenty-seventh inch to the sixty-third inch would be the standard length of the yard. This was decided by the Secretary of the Treasury and the Coast Survey group. This was not adopted by Congress immediately but soon became the accepted idea. Our common system of today was established by use of this length. In 1866 the metric system was legalized, and equivalents between the units of the two systems were calculated. This system is commonly used by those engaged in scientific work, or when a high degree of accuracy is necessary.

Since the legalization of the metric system, there have been, from time to time, legislative attempts to do away with the common system and adopt the metric entirely. It is much simpler and easier to calculate, and more accurate. World War I seems to have done much to produce universal interest in the metric system. In fact, at the present time, Great Britain and the United States are quite alone in not having adopted the metric system as their system of weights.

In general the United States measures are a little less than the British. The units of yard and the pound and their divisions are very nearly the same; but the bushel, gallon, quart, and fluid ounce are quite different.

United States		*Great Britain*
gallon	231 cu. in.	10 lb. water 277.274 cu. in.
quart	$^1/_4$ gallon	$^1/_4$ gallon.
ounce	$^1/_{32}$ quart	

Thus, the British gallon is twenty percent greater than the United States gallon. But this same relation does not hold in the rest of the table as the ounce measure is different. Each system has four quarts per gallon and eight drams per ounce. But the British divide their quart into forty ounces instead of sixteen. Hence, the British gallon and quart are larger, but their ounce and dram smaller. This may explain some of the difficulty in transporting bottled goods from one country to another. The same term is used, but with quite a different meaning. The British ounce is 4% less than that of United States; when buying by the ounce we get less in England, but when buying by the gallon or quart, we get more.

The English bushel, eight gallons, is 3.2% larger than the American, a difference of about seventy cubic inches. The British use the same

quart measure for either dry or liquid materials. The United States liquid quart is more than 20% smaller than the British.

The English ton is larger than the American, since the British take the hundredweight to be one hundred and twelve pounds, and their ton 2240 pounds. The United States uses one hundred pounds, making the ton equal to 2000 pounds. The British ton is some times used by United States in weighing coal or in industry, so the distinction of long and short ton has been made, the long ton weighing 2240 pounds, and the short ton 2000 pounds.

From this study of the units through the years, we may conclude that the tendency seems to have been in the direction of simplification so as to make the tables easier to learn and easier to use.

The development of the table of time is also very interesting. There have not been as many different tables set up in this as there have been in the two former measurements.

Before a definite calendar was developed, people remembered events as being so many winters ago, so many snows ago, or by some fixed season. Even today among the Swedish peasants, a birthday may fall on "Rye harvest" or "Potato harvest". In measuring shorter lengths of time, the number of nights was the measure, hence the term fortnight, the abbreviation for fourteen nights or sennight for seven nights.

The succession of moons came to be the first continuous time measurer known for time periods within a year. There were, it was observed, twelve moons in one year. This was later developed into the lunar solar year. Many years before any conception of a year was known, the primitive peoples had observed that certain seasons rotated and that every twelve moons the same season returned. Gradually each moon gained a name, usually drawn from some physical characterisitic of the earth at that moon. The Greeks worked for many years to form a year based on the moon but were never successful, since the lunar year was about $11\frac{1}{4}$ days short of the real year.

The Egyptians, in the fifth century B. C. had observed and discovered that the year contained $365\frac{1}{4}$ days. But the Greeks who had received their lunar year from the Babylonians, never could understand the Egyptian calendar, and so never adopted it. The orthodox Jews still use the lunar calendar and add an extra month every two or three years. The Egyptian system was known as the steller or solar year. The discovery that there was such a thing as a year and the determination of its length took a great many years of careful study, and dates back to prehistoric times. At first it was thought that 360 days made up a year. As early as 4000 B. C., we find the 360 days year was divided into 36 decades of ten days each, for grouping

the constellations along the celestial equator. This incidentally is the oldest appearance of dividing a circle into 360 divisions, the basis of our degree.

After observing that there are 365 days in the year, they divided it into twelve different periods of thirty days each, the months. Five days of each year were noncalendrical and set aside for religious worship and festivities. Stars determined certain dates of the year. For instance the beginning of the year was set at the first heliacal rising of Serius. This calendar contained the three agricultural peasant seasons, each season four months in length.

This calendar remained in possession of the Egyptians for 3500 years. The Greeks, meanwhile, were trying to perfect the lunar year. Finally in 46 B. C., Julius Cæsar introduced the Egyptian calendar into the Roman empire. But he provided for the addition of one day in every four years, introducing what is known as leap year. This calendar, known as the Julian calendar, was the approved time measure until 1582. In this year Pope Gregory VII discovered that the $365\frac{1}{4}$ day year was too long by 1/300 of a day. To correct this, a provision was made that there should be only ninety seven leap years in four hundred years.

So far all of our time divisions mentioned have been suggested by some natural phenomena; but the week was purely original. The number of days in a week has varied greatly. Regardless of its length it has been of very little use in time measurement, other than a subdivisions of the month. The day is rather a fixed division, but the divisions of the day have been arbitrary. The Egyptians divided the day into twelve parts and night into six parts. The idea of a twenty-four hour day came to Greece from the Egyptians at the time of Alexander the Great. These divisions as well as minutes and seconds, were developed with the introduction of different types of clocks at a much later date.

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The Teachers' Department

Edited by
WM. L. SCHAAF, JOSEPH SEIDLIN, L. J. ADAMS, C. N. SHUSTER

The Most Powerful Thing in the World

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What is the most powerful thing in the world today? Is it a "blockbuster" filled with several tons of concentrated death? Is it the biggest gun on the biggest battleship in the world? Is it the latest rocket bomb, propelled at a speed faster than sound? It could be none of these if the bomb utilizing subatomic energy has actually been developed in the secret laboratory of one of the nations at war. But even if the "atom c bomb" is a reality at this moment, it is laughably puny and impotent compared with the all-engulfing power of the most powerful single thing in the world today.

That thing is very old. It is a thing that children in the grades can have fun with. It is something that every one able to write has handled countless times. Yet it took a genius to invent it for the benefit of the human race, which had been groping for it many hundred years. It seems that this genius lived in India a long time ago, but the circumstances of his life are shrouded in obscurity. His invention has now become the most powerful thing in the world.

In the fiscal year 1940, which was the latest peacetime fiscal year, the expenditures of the United States Government were $9,665,085,539. Rounded off to one significant figure this number is $10,000,000,000. Government expenditures for the fiscal year 1944 were $93, 743, 514, 864, or $100,000,000,000, if similarly rounded off. So one can say that the difference between the yearly economy of the United States at peace and her economy at war is caused by a single cipher affixed to a number of eleven digits.*

What immeasurable and incalculable consequences have flowed from that tiny and unpretentious symbol for zero! By the original impulse of this one cipher all numbers connected with public and private finance have been pushed out of place; the cipher which is the

prime cause reflects itself in other ciphers which have been added to important numbers. The net receipts of the Federal Government have risen from four to forty billion dollars between 1936 and 1944, the deficit from five to fifty billion between 1936 and 1944, the National Debt from twenty billion in 1933 to two hundred billion in 1944, all numbers rounded off to one significant figure.

However, the purely financial consequences are small compared with the social and historical results produced by the crucial zero. The mighty juggernaut of war now rolling over the Axis has been created by it in a large measure. Wherever American materiél is being used on the global battlefronts, it owes its existence to the fact that the American Congress shifted its appropriation figures one place to the left. The tide of war, dangerously flowing in favor of the Axis, was turned, and a threatened Axis triumph has become a crushing defeat. The course of human history has been channeled in a certain direction, and there can be but few among the two billion odd human beings inhabiting this globe whose lives have not been affected in one way or another by the outcome of the war. In our own country new financial, economic, and social problems are in the making, the solution of which will tax the ingenuity and courage of the post-war generation. And all of it stems from a most simple arithmetical operation, which to the untutored mind must seem quite inconsequential. There is dynamite inside that little cipher in the twelfth place, which Webster paradoxically defines as "a character or symbol denoting the absence of all magnitude or quantity howsoever small." Indisputably it is the most powerful single thing on the globe today.

The full realization of its impact on our lives cannot but fill us with awe. Its omnipotence is terrifying. And there are certain conclusions that are forced upon us by this display of might. First, we cannot fail to see that human life on this planet has reached a stupendously high degree of interdependence and collectivity since one little symbol pertaining to the budget of a nation could influence so many lives in so drastic a way. It is not only "one world" we live in, but it appears at this moment that this world is, to a remarkable and unprecendented extent, subject to the central control exercised by one thing. As if the pyramid of humanity were resting on its pin-point vertex, and as a support for the latter we should find—a naught!

Furthermore, the astounding power emanates from a symbol. This fact leads to a second conclusion. A symbol is abstract, intangible; something that cannot be defined by direct reference to the senses as it derives its meaning from an abstract mental process. All the concrete, tangible arsenal of global war, so far as contributed by the United States, is called into being by the intangible, abstract symbol.

The spirit is reigning over matter. Here we have an evidence for the essential spirituality of life.

Lastly, the symbol is a mathematical symbol. After centuries of use it now has come to an unexpected fruition. If the global power of the symbol demonstrates the "oneness" of humanity in space, the the history of the symbol proves the oneness of human civilization in time. From its insignicant beginnings in India it has passed through the vicissitudes of human progress with an unwavering continuity till today it has attained the apex of its influence. So universal and uncanny is this influence that one is reminded of the cult of the Pythagoreans which endowed abstract number with a divine quality. If Sir James Jeans was led by astronomical considerations to suggest that God might well be conceived of as a mathematician, the curious social effects of the omnipotent zero tend to support the same idea. Since the paramount, the most decisive, act in the history of the Twentieth Century can be summed up in a mathemcal symbol, one is tempted to paraphrase the words which allegedly appeared in the sky as Constantine the Great, Roman emperor, went into battle: *In hoc signo vinces*. In the sign of the zero thou shalt win!

Mathematics As a Field of Specialization for College Students

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Introduction. The average college student who is majoring in mathematics has little knowledge of the breadth of the field in which he is working or of the vocational opportunities available to him as a specialist. As a matter of fact, the student who has graduated often has a very limited field of usefulness, not because of the narrowness of mathematics as a subject, but because of the limited amount of mathematics which is usually taught in American colleges today. I refer to the tradition of devoting the large part of our efforts in both teaching and research to so-called pure mathematics and to the exclusion of the many phases of applied mathematics. It is the knowledge of applied mathematics which would fit our students to take responsible jobs in industry and business. This will be discussed more fully later. This paper proposes to discuss some of the relations of mathematics to life in general, to the professions, and to a career It will also deal with some of the problems of mathematics majors in training for and obtaining jobs using their best skills. These students need more specific advice than we, as mathematics teachers, have been able to give them. They need more flexibility in the choice of courses they take for a major if they are to be qualified to enter industry and business and use their skills as mathematicians to best advantage. The members of mathematics departments need more detailed knowledge of the vocational possibilities open to mathematics majors and a systematic knowledge of what mathematical (and other) skills are necessary in these jobs so that they can advise students and provide the necessary courses. Hence this paper will also discuss the vocational possibilities in the field of mathematics, the training in mathematics needed for these vocations and methods of acquainting students with these facts.

Mathematics and Life. A knowledge of mathematics implies a real understanding and greater intellectual independence. Anything less than this is not true mathematical education. But little knowledge in any field can be attained by indirect means. Actual contact with the content of a subject is necessary. This is true also of mathematics. Whether it is to be used as a professional tool in the study

of science, engineering, or the various branches of industry, or as one of the elements of a Liberal Education or for the mere pleasure of learning (there are such motives), it is necessary to have direct contact with the vital living mathematics of the past and present. For mathematics with its rich and venerable past is a vital force in the present day world.

The subject in its various ramifications touches and continues to influence the life of every person who lives in civilized society. The engineer, the soldier and the scientist find it an indispensable tool for the work they must do. So also do workers in business, industry, and commerce. It is becoming increasingly necessary for workers in all these fields to have advanced skills and knowledge of mathematics in order to do their work efficiently. People in ordinary walks of life need fewer skills and less specific knowledge, but even for the humblest person there are certain minimum mathematical skills and concepts without which he can not live fully and efficiently. In addition to our professional and business transactions, mathematics plays an important role for all of us. We are all blessed by the advantages and machines it has made possible. We all use clocks, radios, airplanes, automobiles, and so on, the invention, manufacture and continuing development of which have been made possible by the skills and concepts of the mathematicians.

Mathematics as a Professional Tool. To receive the benefits from most of the inventions which mathematics has made possible does not require a knowledge of the skills of mathematics. However, to work at many professions of the modern world does require such skills. Engineering comes first to mind. The professional engineering work studied in college is preceded by still other mathematics courses (a minimum of 18 college hours), and one cannot finish the engineering course without credit in these courses. While some students acquire the credit without understanding the basic principles and some practicing engineers profess that they depend on a "Handbook" for all the mathematics they use, the persons who must solve the original problems which an engineer meets are the ones who have a fundamental understanding of mathematics and its application to physical problems.

Students of the sciences, particularly the physical sciences, also require a definite set of mathematical skills varying with the subject studied. The same is true of students of business, the social sciences, agriculture and the like. While the skills needed in these fields are not so many or so detailed as those used by scientists and engineers, they are nevertheless essential for a fundamental understanding of the problems. For students who expect to do advanced work in most

of these fields, it is becoming increasingly necessary for them to have a knowledge of the calculus or beyond. Calculus forms the basis on on which most of the applied advanced mathematics is built. In this sense it is essentially an elementary course.

Americans do not like to think of war as a profession to which a large number of our young men look forward. However, such is the case, at least for the present. Mathematics is playing a vital part in the preparation of the modern soldier. This is especially true of the positions of responsibility in all branches of the service. For example the Navy Educational Program lists Mathematics and Science as the basic requirement for all technical work. All officers and prospective officers of the Navy are required to have a year of mathematics as a minimum. The same is true of many branches of the Army.

Mathematics and the Liberal Arts. Mathematics has long been considered a disciplinary subject. At the same time its disciplinary values have been discredited by those who have not or could not learn the subject. This is true in general: the discipline attained by a difficult subject or piece of work is most loudly disclaimed by the persons who have not had that experience. There seems to be no question among informed people that the study of mathematics may furnish a definite discipline to the learner, that is, enable him to do other things better because of the training he has acquired by the study of mathematics. However, the mathematician familiar with the principles of psychology is the first to admit that such training does not necessarily transfer. We know of many many instances where it has not. Numerous mathematicians, scientists and scholars in all fields exhibit their lack of discipline by making absurd statements concerning some of the other fields of knowledge about which they are not qualified to speak. Hence, mathematics may be placed among the Liberal Arts because it is possible to attain discipline by its study. It is true that this discipline is most likely to be attained if the subject is taught with that goal in mind and it is also true that a person may be a well-trained, careful thinker in the field of mathematics (as well as any other field) without exhibiting these qualities in his general thinking.

Mathematics should also be placed among the Liberal Arts because it is one of the forces or influences which has has much to do with the development of modern civilization. It is true that in most instances the beginnings of mathematics is lost in the forgotten history of the past, but it is easy to see that much of modern civilization would have been impossible without the growing knowledge of mathematics to support it. No one will ever know, of course, whether the growing

civilization demanded the invention of mathematical concepts or whether man was enabled to develop the arts of civilization because he had already invented mathematical concepts. We do know, however, that no tribe has ever developed to any appreciable extent without the use of numbers, for example. Others with no apparent advantages except the knowledge of a number system have been able to develop civilizations adequate for their needs. In more recent times it is easy to see that even simple arithmetical calculations, like multiplication or division, were extremely difficult without the use of the Hindu-Arabic number system which was first introduced into Europe in the 12th Century A. D. Even the smallest businesses which we have today would be impossible without this tool. Illustrations like those above could be multiplied in industry, technology, engineering, and the like, indefinitely. A force which has exerted and continues to exert such an influence on the life of man should be familiar to the well educated person. The skills, except the fundamental ones, are not as important as the meanings and applications.

Mathematics As a Career. Persons who apply mathematical knowledge to a particular field are often thought of as physicists, engineers, statisticians, and the like. This is unfortunate and is a situation for which mathematicians as a whole are largely responsible. During the past fifty years mathematical research has been largely divorced from the practical problems of science and engineering. More and more emphasis has been placed on the foundations of mathematics and mathematical research has become to a large extent an exercise in pure logic. It is not implied here that this sort of research is not worthwhile or that it has no practical value, but it is not the type of training which fits a student to go out and earn a living except in a very limited range of activities. Such specialists are known as pure mathematicians, and many of them take pride in the fact that their research has no practical value whatsoever.

However, it is being recognized that workers trained as engineers, physicists, etc., are not wholly qualified to solve the increasingly difficult mathematical problems which occur in industry and business. Hence, there is justification in using the term Applied Mathematician to cover a rather wide field of mathematical workers who range from the highly trained mathematical consultant employed in large industries like the Bell Telephone Company, The Frankfort Arsenal, etc., to the mathematical worker who uses the principles of statistics or the mathematics of finance in many of our smaller business concerns and offices. The different sorts of jobs will be more fully discussed in the next section.

Vocational Possibilities in the Field of Mathematics. We all know that jobs which demand varying amounts of mathematical knowledge exist. In addition to positions as teachers of mathematics in the secondary schools and colleges, there are many positions in business, industry and government service which call for the skills and knowledge of the mathematician. However, few teachers of mathematics in college have any systematic knowledge of these jobs, the specific training required for them or how a student would go about getting one of them. In our institution we have made an attempt to discover the vocational possibilities available for mathematics students. While this survey is by no means complete, we feel sure there are jobs for mathematicians at several levels. The following five may be cited:

1. Positions in business or government offices which involve only elementary mathematics and computations, checking accounts, making statistical computations, and the like, (These positions are often called arithmetical clerks, junior engineers, etc.)
2. Positions in industry and also government service which require the use of much of the mathematics studied by the undergraduate major. (Examples are engineers' helpers in industry, mathematical clerks in the hydrographic office and the Naval Observatory, junior statisticians, and the like.)
3. Positions as marine and air navigators to supply our large merchant marine and future air lines.
4. Positions as teachers of mathematics, applied mathematics and engineering mathematics. (Many of these will start as secondary school teachers.)
5. Positions for a few highly trained applied mathematicians as consultants and research workers (either in industries or in universities) and teachers of advanced courses in the fields of pure and applied mathematics.

Effective ways of finding out about these jobs include interviewing personnel directors or representatives of industrial concerns, use of contacts and data which is often available in the files of the schools of engineering and commerce, seeking the help of former students and acquaintances who may be employed in business concerns, industries or government service and seeking contacts with industry by attending meetings of engineering societies and other groups where company representatives will be present. If summer jobs in industry could be obtained by regular teachers in the mathematics departments, this would give an insight into the nature of the jobs, the mathematical training required, etc.

Training Required. The training in mathematics for these various sorts of jobs must be varied and flexible. Hence in most colleges it will be necessary to broaden considerably the curriculum in mathematics. This will not mean, for the ordinary college, any new courses in pure mathematics. Most colleges have adequate offering in that

field. Probably the situation can be met best by offering a major in Applied Mathematics to cover the fields of statistics, actuarial science, the mathematics of finance, etc., as well as the customary applied mechanics. In this way the requirements for a degree with a major in Applied Mathematics could be made quite flexible and students could be given courses which would qualify them to hold the various jobs which are found to exist.

The following outline shows the general idea of the scheme suggested above. Eventually it could be presented in more detail if it is found that a student entering a given field must have certain specific courses.

Training in Mathematics

I. *Basic Courses*
Algebra, trigonometry, analytical geometry, differential and integral calculus—17 semester hours.

II. *Advanced Curricula*
- A. Pure Mathematics.
  1. Courses in Mathematics
     3 hours each in algebra, geometry and advanced calculus and 3 hours in electives—12 hours.
  2. Minors in any field.
  3. Electives.
- B. Applied Mathematics.
  1. Courses in Mathematics.
     3 hours in advanced algebra and 9 hours of electives in advanced and applied mathematics approved by the student's advisor and the head of the Department of Mathematics—12 hours.
  2. Minor.
     In some related field as engineering, commerce, physical science, biological science, agriculture, etc.
  3. Electives.
     From other related fields.

Advice to the Student. The student should have a full opportunity to know the vocational possibilities in the field of mathematics. This information should not be hard to obtain and the following methods of getting it to the student are proposed.

1. At least one member of the Department of Mathematics should be competent to advise the student in each of the fields of applied mathematics. This professor should be thoroughly familiar with the use of mathematics in this field, the courses required for proper training, the companies, industries and government offices which make use of such students and, if possible, he should have had some practical experience in this type of work.

2. Sample programs should be available, possible in the college catalog, showing prerequisites, sequences of courses, possibly advanced work, desirable minors, and the like.
3. A statement giving the intellectual prerequisites, the training demanded and vocational opportunities for the student of mathematics should be made available to the prospective student in the catalog.
4. A more detailed statement in the form of a pamphlet or brochure containing the data listed in (3), as well as other material, might be prepared to pass out to interested students during interviews either on the campus or in their high schools.

Mathematics thus conceived becomes a living force in our life today. Our students have open to them a vast field of usefulness. They have a wide choice of positions from which they can choose their work. In almost any field the student chooses there is ample advanced work to satisfy the best intellects. The few highly trained applied mathematicians in any of these fields have open to them positions of responsibility as consultants and research workers which compare favorably with any profession, Hence, students who choose these fields can look forward to wide opportunities and the positions available to them will be limited only by their ability and willingness to improve themselves.

Brief Notes and Comments

Edited by
Marion E. Stark

13. *Stumbling Upon Laws.* It has always seemed to the writer that the customary practice with respect to many of our basic laws of mathematics stifles rather than stimulates the attitude of discovering new truths in general form.

Consider for example the Sine and Cosine Laws for the solution of triangles. Instead of saying, "This is the Law of Sines, and that is the Law of Cosines; now let us verify them", why can't we say, "This triangle is not right but oblique, with A, B, and a given. How find b ?"

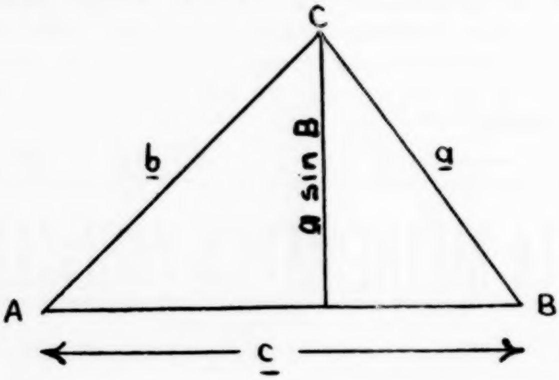

FIG. 1.

The answers to this and the few necessary questions following come, after varying intervals of time to be sure, but *always come* from the class, which, of course, has been trained from the beginning to evaluate on sight any side of a right triangle in terms of another side and an acute angle.

A. Draw an altitude.
Q. Any difference which altitude ?
A. Yes, don't cut up the given a or required b.
Q. How long is the altitude ?
A. $a \sin B$.
Q. Then how long is b ?
A. $b = a \sin B \csc A$.

After calling attention to a different form:

$$\frac{b}{a}=\frac{\sin B}{\sin A},$$

and finding similar results from other experiments, including the "Ambiguous Case" when a, b, and A are given, it is easy to say, "We have blundered upon a very important law called the Law of Sines. But if a, c, and B were given would the second right triangle have yielded to this treatment ?"

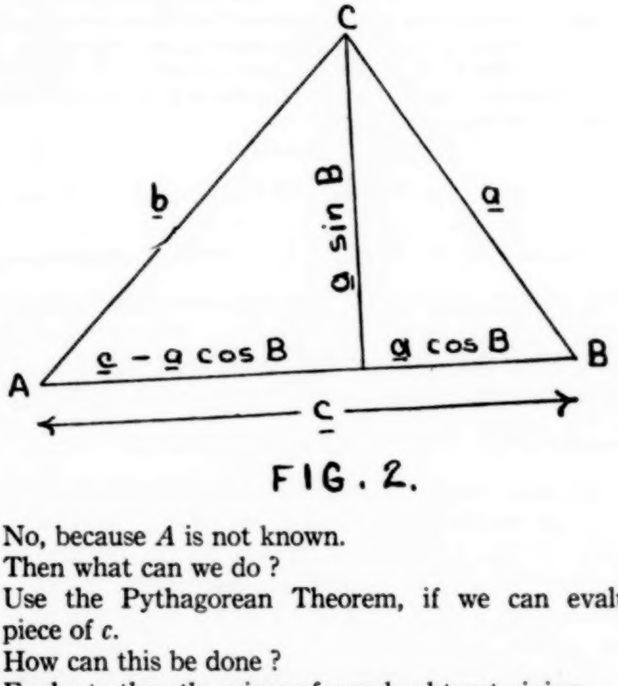

FIG. 2.

A. No, because A is not known.
Q. Then what can we do ?
A. Use the Pythagorean Theorem, if we can evaluate that piece of c.
Q. How can this be done ?
A. Evaluate the *other* piece of c and subtract giving

$$b^2=(a \sin B)^2+(c-a \cos B)^2$$
$$=a^2 \sin^2 B+c^2-2ca \cos B+a^2 \cos^2 B.$$

Q. Any chance of simplifying ?
A. Yes. $a^2 \sin b^2 B+a^2 \cos^2 B=a^2(1)=a^2$. Hence $b^2=a^2+c^2-2ac \cos B$.
Q. Could this be used to find B if a, b, and c were given ?
A. Yes.

The teacher can now explain that we have blundered upon the Law of Cosines and that these two laws suffice for the solution of *all* triangles.

Chauncy Hall School, Boston. RAY D. FARNSWORTH.

Problem Department

Edited by
E. P. STARKE and N. A. COURT

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to E. P. STARKE, Rutgers University, New Brunswick, N. J.

SOLUTIONS

No. 547. Proposed by *N. A. Court*, University of Oklahoma.

Consider the tetrahedron (L) formed by the polar planes of a given point M with respect to four given spheres. The midpoints of the segments joining M to the vertices of (L) form a tetrahedron orthological to the tetrahedron determined by the centers of the given spheres.

Solution by *P. D. Thomas*, RT3c, U. S. Navy.

Since the polar planes are respectively perpendicular to the lines joining M to the centers of the four given spheres, it is clear that (L) is orthological to the tetrahedron formed by the centers of the four given spheres.* Now the tetrahedron formed by joining the midpoints of the segments from M to the vertices of (L) will have its faces respectively parallel to those of (L), hence it will also be orthological to the tetrahedron formed by the centers of the given spheres.

Also solved by *J. S. Guérin*.

EDITORIAL NOTE. The point P common to the polar planes of M with respect to the three given spheres (B), (C), (D) is conjugate with respect to each of these spheres, therefore the sphere (MP) having the segment PM for diameter is orthogonal to each of the three spheres considered, hence the mid-point P_0 of PM lies on the radical axis p of those three spheres, which line is perpendcular to the plane of centers BCD of the spheres (B), (C), (D), and passes through the radical center U of the four given spheres (A), (B), (C), (D).

*N. A. Court, *Modern Pure Solid Geometry*, p. 147, No. 460. Macmillan, 1935.

Similarly for the analogous lines q, r, s passing through the mid-points Q_0, R_0, S_0 of the analogous segments MQ, MR, MS. The four perpendiculars thus have the point U in common.

The proposition may be generalized. Let P', Q', R', S' be the points which divide the segments MP, MQ, MR, MS in the same ratio, say k. The perpendiculars p', q', r', s' from the points P', Q', R', S' to the respective faces of the tetrahedron $ABCD$ correspond to the perpendiculars p, q, r, s in a homothecy (M, k) having M for center and k for homothetic ratio, hence p', q', r', s' have a point U' in common and U' is collinear with M and U.

Furthermore, if the value of the ratio k is made to vary, the point U' will describe a straight line, namely the line MU joining the given point M to the radical center U of the given four spheres.—N. A. C.

No. 554. Proposed by *N. A. Court*, University of Oklahoma.

The two lines which join the circumcenter of a triangle and its isotomic conjugate to the Lemoine point and the orthocenter, respectively, are parallel.

I. Solution by *Nev. R. Mind.*

The circumcircle (O) of a triangle $(T)=ABC$ and the circle (AP) having for diameter the cevian AOP of (T) passing through the circumcenter O of (T) have for radical axis the tangent to (O) at the vertex A, which tangent passes through the point of intersection of the side BC with the Lemoine axis of (T) (see the proposer's *College Geometry*, p. 229). Similarly for the cevians BOQ and COR.

Consequently: *The pole of the Lemoine axis with respect to the polar of* (T) *coincides with the isotomic* O', *with respect to* (T), *of the circumcenter* O *of* (T) (N. A. Court, *On the cevians of a triangle*, this MAGAZINE, Vol. 18, No. 1, October, 1943, p. 4, Art. 3b).

Thus the line $O'H$ joining O' to the orthocenter H of (T), which point coincides with the center of the polar circle of (T), is perpendicular to the Lemoine axis. But so is also the Brocard diameter of (T), which line passes both through O and the Lemoine point of (T). Hence the proposition.

II. Solution by *Howard Eves*, Syracuse University.

Let us designate the circumcenter by O, its isotomic by T, the Lemoine point by L, and the orthocenter by H. Then the homogeneous areal coordinates of these points are

$$O : (a \cos A,\, b \cos B,\, c \cos C),$$
$$T : \big(1/(a \cos A),\, 1/(b \cos B),\, 1/(c \cos C)\big),$$

$$L : (a^2, b^2, c^2),$$

$$H : (a/\cos A, b/\cos B, c/\cos C).$$

The coefficients of line OL are then proportional to

$$b^2c \cos C - bc^2\cos B,\ c^2a \cos A - ca^2\cos C,\ a^2b \cos B - ab^2\cos A,$$

or, using the law of cosines to eliminate $\cos A$, $\cos B$, $\cos C$, to

$$b^2c^2(b^2 - c^2),\ c^2a^2(c^2 - a^2),\ a^2b^2(a^2 - b^2).$$

The coefficients of line TH are proportional to

$$(ab^2 - ac^2)\cos A,\ (bc^2 - ba^2)\cos B,\ (ca^2 - cb^2)\cos C,$$

or, using the law of cosines, to

$$-a^2b^4 + a^4b^2 + a^2c^4 - a^4c^2,\ -b^2c^4 + b^4c^2 + b^2a^4 - b^4a^2,$$
$$-c^2a^4 + c^4a^2 + c^2b^4 - c^4b^2.$$

We now form the determinant

$$\begin{vmatrix} b^2c^2(b^2 - c^2) & \text{etc.} & \text{etc.} \\ -a^2b^4 + a^4b^2 + a^2c^4 - a^4c^2 & \text{etc.} & \text{etc.} \\ 1 & 1 & 1 \end{vmatrix}$$

Since the sum of the first two rows of this determinant is equal to the third row multiplied by $a^4(b^2 - c^2) + b^4(c^2 - a^2) + c^4(a - b^2)$, it follows that the determinant vanishes. But the vanishing of this determinant is the condition that OL is parallel to TH. Hence the theorem.

No. 560. Proposed by *N. A. Court*, University of Oklahoma.

A plane passing through a given point F cuts the lines DA, DB, DC in the points P, Q, R, and the six points P, Q, R, A, B, C lie on a sphere (S). If the plane ABC varies, remaining parallel to itself, what will be the locus of the center of the sphere (S) ?

Solution by *Howard Eves*, Syracuse University.

Consider the oblique cone having D for vertex and circle ABC for base. Through F pass a plane antiparallel to plane ABC and cutting DA, DB, DC in P, Q, R respectively. Then, as plane ABC varies, remaining parallel to itself, the six points A, B, C, P, Q, R are always cospherical. The locus of the center of this sphere is, then, the straight line perpendicular to plane PQR at the circumcenter of triangle PQR. (see Chap. VI in N. A. Court's *Modern Pure Solid Geometry.*)

Also solved by *J. S. Guérin*.

BIBLIOGRAPHICAL NOTE. The corresponding question in the plane was considered in *Bulletin des sciences mathématique et physique élémentaires*, Vol. 4, 1898-1899, p. 43, Q. 849.—N. A. C.

No. 563. Proposed by *E. P. Starke*, Rutgers University.

On the floor ABC of a triangular room is placed a rug with its edges parallel to the sides of the room. Determine the largest such rug which can be turned entirely around on the floor without the walls interfering.

I. Solution by *Howard Eves*, Syracuse University.

In the following, numbers in parentheses refer to articles in Johnson's *Modern Geometry*.

We will call a triangle, $A'B'C'$, whose vertices lie on each side of triangle ABC, an inscribed triangle of ABC. If A', B', C' lie on the sides unproduced of ABC, then $A'B'C'$ will be called a proper inscribed triangle; otherwise an improper inscribed triangle.

Let us generalize the given problem somewhat at the outset by considering the problem of finding the largest permissible triangular rug *of any given species* $A'B'C'$. We observe that this problem is allied to that of finding the smallest triangle directly similar to $A'B'C$ and inscribed in triangle ABC. If this smallest inscribed triangle is properly inscribed then it is our solution. If, on the other hand, this smallest triangle is improperly inscribed, then it is readily seen that our solution is a triangle of the given species having its longest side equal to the shortest altitude of ABC.

Now all triangles directly similar to $A'B'C'$ and having homologous vertices on respective sides of ABC have the same Miquel point, P, which is their common center of similitude (188, cor. b). Of these triangles the pedal triangle of point P with regard to ABC is the minimum one, and its area is $(R^2-OP^2)K/R^2$, where R is the circumradius, O the circumcenter, and K the area of ABC (198). Now there are three possible positions for the point P, according as A' is on side BC, CA, or AB. We select the pedal triangle of that position of P which is most remote from O. If, then, this selected pedal triangle is properly inscribed in ABC, we choose it for our rug, and its area is given by the above formula for the chosen P. If this pedal triangle is improperly inscribed in ABC, we choose for our rug a triangle of the given species having its longest side equal to the shortest altitude of ABC.

If, now, we take $A'B'C'$ directly similar to ABC (as the printed problem requires), then the three possible positions for P are the cir-

cumcenter and the two Brocard points of ABC (190, cor. b and 442). But the pedal triangles of the two Brocard points are congruent (441, second theorem), and their ratio of similitude to triangle ABC is $\sin\omega$, where ω is the Brocard angle of ABC (444). If, therefore, the pedal triangles of the Brocard points are properly inscribed in ABC, we select one of these triangles for our rug, and its area is $K\sin^2\omega$. If they are improperly inscribed in ABC we select for our rug a triangle directly similar to ABC and having area $4K^3/c^4$, where c is the longest side of ABC.

II. Solution by the *Proposer*.

Let the rug $A'B'C'$, directly similar to the room ABC, have its vertices A', B', C', respectively, on sides c, a, b of the room, and let Θ be the inclination of c' to c. The ratio of similitude, $\lambda = c'/c$ is easily found to satisfy.

$$1/\lambda = \cos\Theta + (\cot A + \cot B + \cot C)\sin\Theta.$$

The minimum of these values of λ corresponds to $d\lambda/d\Theta = 0$, which gives $\tan\Theta = \cot A + \cot B + \cot C$, whence

$$\lambda = [1 + (\cot A + \cot B + \cot C)^2]^{-\frac{1}{2}} = \cos\Theta$$

$$= 2K(a^2b^2 + a^2c^2 + b^2c^2)^{-\frac{1}{2}} = \sin\omega,$$

where K is the area of the floor and ω is the Brocard angle.

The same minimum λ is obtained when A', B', C' are placed respectively on b, c, a of ABC. If A', B', C' are on a, b, c, respectively, we find easily $\lambda = \frac{1}{2}\sec\Theta$, and its minimum value of $\frac{1}{2}$ occurs when corresponding sides are parallel.

Now in rotating the rug we may have a critical position when (1) the three vertices of the rug are at the three walls of the room, or (2) the vertex of the rug is at a corner of the room and the more remote of the other vertices is at the opposite wall. For type (1) the above paragraphs give the extreme values of λ. For type (2) we need only note that the longest side of the rug cannot exceed the shortest altitude of the room: thus, if $a \leqq b \leqq c$, we have $\lambda \leqq a\sin B/c = 2K/c^2$.

Finally λ for the largest acceptable rug must be the smallest of the numbers $\sin\omega$, $2K/c^2$, $\frac{1}{2}$, from which $\frac{1}{2}$ may be dropped since it is known that $\sin\omega \leqq \frac{1}{2}$.

III. Solution by *W. B. Carver*, Cornell University.

Consider the rug $A'B'C'$ $(A' \leqq B' \leqq C')$ placed on the floor ABC so that angles C' and C coincide. If now the rug is turned so as to be always as close to corner C as possible (i. e. one vertex at C or one vertex each on CA and CB) the loci of the remaining vertices of the rug

in the various positions comprise several arcs of ellipses and circles. Obtain the equations of these loci with reference to an oblique Cartesian system having CA and CB for its axes. Then the smallest ratio of similitude λ is determined such that side AB does not cross any locus but touches at least one. The algebraic geometry used to carry out this program is rather heavy and laborious, but entirely straightforward.

In addition to results noted in I and II the following facts are obtained. For $\cos c \geqq 0$, $\lambda = \sin \omega$ is the value to use;

for $\cos C \leqq (1-\sqrt{2})/2$, $\lambda = 2K/c^2$;

for $(1-\sqrt{2})/2 < \cos C < 0$,

further data are needed to determine which value is to be used. In case $\sin \omega = 2K/c^2$, a curious situation arises. As the maximum rug is turned on the floor it passes through four critical positions where it would not get by if it were any larger. It is amusing to find these four critical positions with a paper model: for example, let the sides of the room be $a=5$, $b=2\sqrt{15}$, $c=10$, and $\lambda=\sqrt{231}/40$.

Also solved by *Pvt. William Leong*.

No. 566. Proposed by *N. A. Court*, University of Oklahoma.

Construct a skew quadrilateral such that one side shall pass through a given point and shall be perpendicular to the opposite side, and the remaining two sides shall lie on two given skew lines.

Solution by *Howard Eves*, Syracuse University.

Construction: Let P be the given point and let a and b be the given skew lines. Construct the planes Pa and Pb to intersect in line c, a third side of the skew quadrilateral. Then, through a and b pass planes parallel to c', any line perpendicular to c, to intersect in d, the fourth side of the skew quadrilateral.

Proof: See Arts. 4 and 5 of N. A. Court's *Modern Pure Geometry*.

Discussion: For a finite skew quadrilateral, P cannot lie on the common perpendicular of a and b, nor on the plane through a parallel to b, nor on the plane through b parallel to a. For in the first case the intersection d is at infinity, and in the other two cases c will be parallel to a or b. Because of the infinite choice for direction of c' it follows that for any other position of P an infinite number of solutions exists.

Also solved by *J. S. Guérin*, *P. D. Thomas*, and the *Proposer*.

No. 570. Proposed by *E. P. Starke*, Rutgers University.

A chain (perfectly flexible and inextensible) hangs with its ends fastened at two points, A, B, in a horizontal line. The lowest point of the chain is p units below the line AB, and its length is s units. Determine the distance AB in terms of p and s.

Solution by *J. S. Guérin*, Catholic University, Washington, D. C.

The equation of the catenary in rectangular coordinates being

$$y=\tfrac{1}{2}a(e^{x/a}+e^{-x/a}), \tag{1}$$

one has at once

$$1+y'^2=\tfrac{1}{4}(e^{2x/a}+e^{-2x/a}+2)=y^2/a^2,$$

and thence

$$\frac{s}{2}=\int_0^x (1+y'^2)^{\frac{1}{2}}dx=\int_0^x \frac{y}{a}dx=\frac{a}{2}(e^{x/a}-e^{-x/a}) \qquad \text{or}$$

$$s=a(e^{x/a}-e^{-x/a}). \tag{2}$$

With $a+p=y$, (1) becomes

$$a+p=\tfrac{1}{2}a(e^{x/a}+e^{-x/a}). \tag{3}$$

From (2) and (3), one obtains

$$2ae^{x/a}=2(a+p)+s, \quad 2ae^{-x/a}=2(a+p)-s. \tag{4}$$

If these are multiplied and solved for a, the result is

$$a=(s^2-4p^2)/8p. \tag{5}$$

From (4), in the form

$$x=a\log\frac{2(a+p)+s}{2a},$$

upon substitution of (5), one has

$$AB=2x=\frac{s^2-4p^2}{4p}\log\frac{s+2p}{s-2p}.$$

Also solved by *Howard Eves*, *H. E. Fettis*, *Frank Hawthorne*, *W. S. Loud*, *A. Sisk*, and *P. D. Thomas*. Hawthorne notes that

$$\log\frac{s+2p}{s-2p}=4\left(\frac{p}{s}+\frac{4}{3}\frac{p^3}{s^3}\right)$$

is a fair approximation when p/s is small. It leads to a simple working formula

$$AB = s - 8p^2/3s^3$$

which is found to be very useful in measuring horizontal distances with a flexible tape, in stringing power lines, etc. See Synge and Griffith, *Principles of Mechanics*, pp. 99-104. See also Reddick, *Differential Equations*, pp. 178-184.

PROPOSALS

No. 591. Proposed by *E. P. Starke*, Rutgers University.

A right cylinder whose cross-section is an ellipse of eccentricity e is placed with its axis horizontal upon an inclined plane, and it does not roll down. What is the greatest possible inclination of the plane? Assume there is no resistance to rolling but that the cylinder cannot slide down the plane.

No. 592. Proposed by *Nev. R. Mind.*

If the chords AP, BQ of a circle meet in M, the perpendicular to AP at P, the perpendicular to BQ at Q, and the perpendicular from M to AB, are concurrent.

No. 593. Proposed by *Howard D. Grossman*, New York City.

An array of a vertical lines at unit intervals all intersecting $b(\leq a)$ horizontal lines at unit intervals contains $b(b-1)(3a^2-9a+2b+2)/12$ non-square rectangles.

No. 594. Proposed by *N. A. Court*, University of Oklahoma.

The base of a variable triangle is fixed and the opposite vertex describes a straight line perpendicular to but not coplanar with the base of the triangle. Find the locus of the feet of the altitudes on the two variables sides of the triangle.

No. 595. Proposed by *Nev. R. Mind.*

If two circles (P), (Q) have their centers on a given circle (A) and are orthogonal to a second given circle (B), the center of the circle of similitude of the circles (P), (Q) lies on the radical axis of the circles (A), (B).

No. 596. Proposed by *Harold S. Grant*, Rutgers University.

Express the equations of the real central conics $x^2/a^2 \pm y^2/b^2 = 1$ parametrically in terms of $A = \frac{1}{2}\int_C(xdy - ydx)$, C comprising the contour composed of the curve and the two radii vectores emanating from the origin to the points $(a,0)$ and (x,y).

Bibliography and Reviews

Edited by
H. A. Simmons and P. K. Smith

Nautical Mathematics and Marine Navigation. By S. A. Walling, J. C. Hill, and C. J. Rees. The Macmillan Company, New York, 1944. ix+221 pages. $2.00.

Nautical Mathematics and Marine Navigation has been designed "for the benefit of those who have become 'rusty' by the lapse of time and as a stimulant to those who found difficulty or distaste in the academic study of mathematics." It recognizes that "a most necessary preliminary to the intelligent understanding of many of the activities and duties of a sailor's life is a sound knowledge of the elementary principles of mathematics and their nautical applications."

The first part of the book is devoted to nautical mathematics and the second part to marine navigation. The nautical mathematics consists of numerous exercises and problems involving the operations of addition, subtraction, multiplication, and division with arithmetic whole numbers and fractions; significant figures; the metric system; measurement; averages; ratio and proportion; percentage; square root; and graphs. Among the topics treated in marine navigation are the principles of navigation, some preliminary geometry, the geometry of the earth and the Mercator chart, dead reckoning, the magnetic compass, estimated position, bearings, vectors, fixes, the sextant and observed position. An appendix includes a hydrographic chart, a table of square roots, and answers to the problems and exercises in the book.

This book should appeal to teachers who want an abundance of arithmetic exercises for drill purposes, a source of interesting arithmetic problems of seamanship, or a brief treatment of marine navigation. The skill with which opportunity has been provided for the gradual learning of concepts of navigation while practice is given in arithmetic is to be commended. However, teachers who desire meanings made clear through detailed explanations (in the exposition and in the illustrative examples) will find the book less satisfactory in this respect.

Northwestern University. Eugene W. Hellmich